REU Day 5
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Networks on surfaces

We'll want networks on them in which each internal vertex looks like this: → · → ,
not on the boundary.
and each boundary vertex looks like this:

\[ \text{source} \quad \text{OR} \quad \text{sink} \]

Internal vertices are given (variable) weights.

**EXAMPLE 1:**

\[
\begin{array}{c}
\text{surface} = \text{disk}
\end{array}
\]
**Example 2:**

Surface = cylinder

**Example 3:**

Surface = torus
No directed contractible cycles are allowed:

Measurements of homotopy classes of walks from sources to sinks, or closed (non-contractible) walks.

Here is how you pick up the variable weights along walks...
"Highway measurements"

"Underway measurements"

distinguishable because the surface is assumed orientable.
**Example 1:**

- **Weight:**\(ab, a+c, b, d, \)\(ad\)
  - 3→8, 3→5, 2→8, 2→6
  - 3→6

- **Weight:**\(cd, a+c, b, d, bc\)
  - 7→4, 2→4, 1→5, 7→5, 1→4

- **Highway**

- **Underway**
EXAMPLE 2:

\[ \text{has weight } abc \]

\[ \text{has weight } ab + ac + bc \]

\[ \text{has weight } a + b + c \]

\[ \frac{a^2 + b^2 + c^2}{a^3 + b^3 + c^3} \]
\[
\begin{align*}
\frac{a^2}{a+b+c+d} + ab + ac + bc
\end{align*}
\]

**Example 3:**

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<tbody>
<tr>
<td>0</td>
<td>dull</td>
<td>(\frac{ab}{cd})</td>
<td>(\frac{a^2b^2}{\frac{c}{d}})</td>
<td>(\ldots)</td>
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<tr>
<td>1</td>
<td>(\frac{ac}{bd})</td>
<td>dull</td>
<td>(bd+ac+ab+cd)</td>
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<tr>
<td>2</td>
<td>(\frac{a^2+bd^2}{2})</td>
<td>(ab+bd)</td>
<td>dull</td>
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<td>3</td>
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CONJECTURE:
In characteristic zero, the algebra generated by the highway measurements is the same as the algebra generated by the underwa measurements.

REU Problem 5a:
Prove this CONJECTURE for an $(n,m,k)$ torus network.

$n$ horizontal cycles
$m$ vertical cycles
$k/n$ Dehn twists before reconnecting
Now for more than one path... 
Say 2 paths are noncrossing if they have no common edges (so a closed path that goes around twice crosses itself!)

The 2nd kind of measurement sums over families of noncrossing paths with fixed overall topology.
Let $a = \# \text{crossings with}$

$b = \# \text{crossings with}$

**Exercise 14**: Determine for which $(a, b)$ a measurement exists at all.
**Exercise 15(a)**: Prove one gets the same measurement here using highway or underway rules.

**Exercise 15(b)**: Prove the measurements of the 1st and 2nd kinds generate the same algebra. Can you give formulas expressing the generators in terms of each other?

**REU Problem 5(c)**: Prove that the non constant measurements of the 2nd kind are algebraically independent.
EXAMPLE: \((n, m, k) = (1, 2, 0)\)

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<tbody>
<tr>
<td>0</td>
<td>(*)</td>
<td>(a+b)</td>
<td>(\frac{a+b^2}{2})</td>
<td>(\frac{a^3+b^3}{3})</td>
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<tr>
<td>1</td>
<td>(ab)</td>
<td>(a+b)</td>
<td>(\ast)</td>
<td>(a+b)</td>
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<tr>
<td>2</td>
<td>(ab^2 + a^2b)</td>
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(1, 2, 0)

2nd kind of measurements

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<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>a0</td>
<td>a+b</td>
<td>1</td>
<td></td>
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<tr>
<td>2</td>
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Note that \( ab, a+b \) generates the same algebra as 
\[ a^3 + b^3, \frac{a^3 + b^3}{2}, \frac{a^3 + b^3}{3}, \ldots \]
which is also the same algebra generated by 
\[ ab, a+b, ab + a^2 b \ldots \]

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The disk case

2nd kind of highway measurement: 
(1,2) \( \rightarrow \) (1',2')
What about 1st kind of measurements?

1 → 1' \quad b+d
1 → 2' \quad be
2 → 1' \quad 1
2 → 2' \quad ce

dc + bc + de = \det \begin{bmatrix} b+d & be \\ 1 & c+e \end{bmatrix}

Lindström Lemma

(Gessel-Viennot method)
EXERCISE 16: Prove this, i.e. when the sources $1,2,...,n$ and sinks $1',2',...,n'$ are separated on the disk boundary.

The 2nd kind boundary measurement $(1,2,...,n) \rightarrow (1',2',...,n') = \det [a_{ij}]$

where $a_{ij}$ is the 1st kind boundary measurement $i \rightarrow j'$. 