REU PROBLEM: NONCROSSING TREE PARTITIONS AND SHARD INTERSECTIONORDERS

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A set partition \( B = (B_1, \ldots, B_r) \) of \([n] := \{1, \ldots, n\}\) is a family of subsets \( B_i \in [n] \) where \( B_i \cap B_j = \emptyset \) and \( \cup_{i=1}^r B_i = [n] \). A partition \( B \) is noncrossing if no two of its blocks have \( i, k \in B_s, j, \ell \in B_t \) s.t. \( i < j < k < \ell \). Let \( \#NC(n) \) denote the number of noncrossing partitions of \( n \).

It is a classical result that \( \#NC(n) = C_n := \frac{1}{n+1} \binom{2n}{n} \). The number \( C_n \) is known as the \( n \)th Catalan number.

1. Lattices

A poset (or partially ordered set) is set \( P \) and a relation \( \leq \) called a partial order that satisfies

- \( x \leq x \),
- if \( x \leq y \) and \( y \leq x \), then \( x = y \), and
- if \( x \leq y \) and \( y \leq z \), then \( x \leq z \)
for any \( x, y, z \in P \). We will write \( P \) instead of \((P, \leq)\), unless it is not clear which partial order on \( P \) is being used.

A subset \( C \) of \( P \) is called a chain if any two elements of \( C \) are comparable (under the partial order on \( P \)). The chain \( C \) is maximal if it is not contained in any larger chain of \( P \).

A lattice \( L \) is a poset where any \( x, y \in L \) have a join and a meet. A join (resp. meet) of \( x \) and \( y \), denoted \( x \lor y \in L \) (resp. \( x \land y \)), must satisfy the following

- \( x, y \leq x \lor y \) (resp. \( x \land y \leq x, y \)) and
- if \( z \in L \) where \( x \leq z \) and \( y \leq z \) (resp. \( z \leq x \land y \)), then \( x \lor y \leq z \) (resp. \( z \leq x \land y \)).

We will only consider finite lattices and posets. All finite lattices have a unique maximal (resp. minimal) element, denoted \( \hat{1} \) (resp \( \hat{0} \)). Ask students why.

2. Shellability

Assume that \( P \) is finite poset, that all maximal chains of \( P \) are of the same length \( r \), and that \( \hat{0}, \hat{1} \in P \) (\( P \) is a finite graded poset). Let \( \text{Cov}(P) := \{(x, y) \in P^2 : x \rightarrow y \text{ in } P\} \) be the set of covering relations of \( P \). A map \( \lambda : \text{Cov}(P) \rightarrow Q \) where \( (Q, \leq_Q) \) is some poset is called an \( n \)th (edge) labeling. A maximal chain \( C = c_1 < \cdots < c_{r+1} \) of \( P \) is increasing if \( \lambda(c_1, c_2) \leq_Q \cdots \leq_Q \lambda(c_r, c_{r+1}) \). Given two maximal chains \( C = c_1 < \cdots < c_{r+1} \) and \( C' = c'_1 < \cdots < c'_{r+1} \) in \( P \), we say \( C \) is lexicographically smaller than \( C' \) if \( (\lambda(c_1, c_2), \ldots, \lambda(c_r, c_{r+1})) \) lexicographically precedes \( (\lambda(c'_1, c'_2), \ldots, \lambda(c'_{r}, c'_{r+1})) \).

Definition 2.1. A labeling \( \lambda : \text{Cov}(P) \rightarrow Q \) is an EL-labeling (or edge lexicographical labeling) of \( P \) if for every interval \( [x, y] := \{z \in P : x \leq z \leq y\} \) of \( P \),

i) there is a unique increasing maximal chain \( C \) in \( [x, y] \), and
ii) \( C \) is lexicographically smaller than any other maximal chain \( C' \) in \( [x, y] \).

If \( P \) admits an EL-labeling, it is said to be EL-shellable.

Theorem 2.2 (Björner). Let \((B, B') \in \text{Cov}(\text{NC}(n))\) and let \( B_i, B_j \in B \) be the blocks that are merged to produce \( B' \). Then the labeling \( \lambda : \text{Cov}(\text{NC}(n)) \rightarrow [n] \) defined by \( \lambda(B, B') := \max\{\min(B_i, \min(B_j)\} \) is an EL-labeling. Thus the lattice \( \text{NC}(n) \) is EL-shellable.
3. Noncrossing Tree Partitions

Let $T$ be a tree embedded in the disk $D^2$ in such a way that a vertex of $T$ lies on the boundary of $D^2$ if and only if that vertex is a leaf of $T$. The tree $T$ has boundary vertices and interior vertices.

The tree $T$ has an important set of subgraphs, which we will call segments. A segment $s = (v_0, \ldots, v_t) = [v_0, v_t]$ with $t \geq 1$ is a sequence of interior vertices of $T$ that turn sharply at $v_i$ for each $1 \leq i \leq t - 1$. A vertex of $T$ is not a segment. Let $\text{Seg}(T)$ denote the set of segments of $T$.

A red admissible curve $\gamma : [0, 1] \to D^2$ for a segment $s = [v_0, v_t]$ is a simple curve where

- its endpoints are $v_0$ and $v_t$,
- $\gamma$ may only intersect edges of $T$ of the form $(v_{i-1}, v_i)$ where $i \in [t]$, and
- $\gamma$ must leave its endpoints “to the right.”

Two segments are noncrossing if they admit red admissible curves that do not intersect.

A noncrossing tree partition $B = (B_1, \ldots, B_k)$ is a set partition of the interior vertices of $T$ where

- there is a (unique) set of segments $\text{Seg}_r(B_i) \subset \text{Seg}(T)$ connecting the vertices in $B_i$ and any two segments in $\text{Seg}_r(B_i)$ may agree only at their endpoints and
- any segments $s_1 \in \text{Seg}_r(B_i)$ and $s_2 \in \text{Seg}_r(B_j)$ are noncrossing.

**Theorem 3.1** (G.–McConville). The set $\text{NCP}(T) := \{\text{noncrossing tree partitions of } T\}$ partially ordered by refinement (i.e. if $B = (B_1, \ldots, B_k) \leq B' = (B_1, \ldots, B'_l)$, then each block $B_i$ is contained in some $B'_j$) is a lattice.

**Exercise 3.2.** Find a tree $T$ where

a) $\#\text{NCP}(T)$ is not equal to any Catalan number

b) $\#\text{NCP}(T)$ is equal to a Catalan number, but $\text{NCP}(T) \not\cong \text{NC}(n)$ for any $n$.

**Problem 3.3.** Let $T$ be a tree embedded in a disk with $n$ interior vertices so that the rank of $\text{NCP}(T)$ is $n - 1$.

a) Show that $\text{NCP}(T)$ is EL-shellable.

b) Find a formula for the number of maximal chains of $\text{NCP}(T)$.

**Remark 3.4.** By Problem 3.3, the simplicial complex $\Delta \left( \overline{\text{NCP}(T)} \right)$ will be homotopy-equivalent to a wedge of $(n - 3)$-dimensional spheres. The number of such spheres will be $\#\{\text{maximal chains of } \text{NCP}(T)\} - 1$.

4. Shard Intersection Order of Biclosed Sets

A tree $T$ defines another lattice whose combinatorics we want to further understand.

Two segments $s_1$ and $s_2$ are composable if $s_1 \circ s_2 \in \text{Seg}(T)$. A set $B \subset \text{Seg}(T)$ is closed if for any composable segments $s_1, s_2 \in B$, one has that $s_1 \circ s_2 \in B$. We say $B$ is biclosed if $B$ and $\text{Seg}(T) \setminus B$ are closed. Let $\text{Bic}(T)$ denote the set of biclosed sets of $T$ partially ordered by inclusion.

We introduced the lattice of noncrossing tree partitions $\text{NCP}(T)$ in order to describe the shard intersection order of $\overline{\text{FG}}(T)$. Now we want to understand the shard intersection order of $\text{Bic}(T)$.

**Exercise 4.1.** Let $B_1, B_2 \in \text{Bic}(T)$.

a) Describe $B_1 \vee B_2$.

b) Use a) to show that $\text{Bic}(T)$ is a lattice.

**Theorem 4.2** (G.–McConville). The lattice $\text{Bic}(T)$ is a congruence-uniform lattice (i.e. it can be constructed from the one element lattice by a finite sequence of interval doublings (this definition is a result of Day)). Also, it is graded by cardinality of biclosed sets.
Proposition 4.3 (essentially Reading). A lattice is congruence-uniform if and only if it admits a CU-labeling.

Definition 4.4. A labeling \( \lambda : \text{Cov}(L) \to Q \) is a CN-labeling if and dual \( L^* \) satisfy the following: For elements \( x, y, z \in L \) with \((z, x) \in \text{Cov}(L) \) and maximal chains \( C_1 \) and \( C_2 \) in \([z, x \lor y] \) with \( x \in C_1 \) and \( y \in C_2 \),

(CN1) the elements \( x' \in C_1, y' \in C_2 \) such that \((x', x \lor y), (y', x \lor y) \in \text{Cov}(L) \) satisfy
\[
\lambda(x', x \lor y) = \lambda(z, y), \quad \lambda(y', x \lor y) = \lambda(z, x);
\]
(CN2) if \((u, v) \in \text{Cov}(C_1) \) with \( z < u, v < x \lor y \), then \( \lambda(z, x), \lambda(z, y) < Q \lambda(u, v) \);
(CN3) the labels on \( \text{Cov}(C_1) \) are pairwise distinct.

We say that \( \lambda \) is a CU-labeling if, in addition, it satisfies
(CU1) for any elements \( j, j' \in L \) that cover unique elements \( j_*, j'_* \in L \), respectively, one has that \( \lambda(j_*, j) \neq \lambda(j'_*, j') \);
(CU2) for any elements \( m, m' \in L \) that are covered by unique elements \( m^*, m'^* \in L \), respectively, one has that \( \lambda(m, m^*) \neq \lambda(m', m'^*), \)

Theorem 4.5 (G.–McConville). The labeling \( \lambda : \text{Cov}(\text{Bic}(T)) \to \text{Seg}(T) \) defined by \( \lambda(B, B \sqcup \{s\}) = s \) is a CN-labeling (here \( \text{Seg}(T) \) has the partial order \( s_1 \leq_{\text{Seg}(T)} s_2 \) if \( s_1 \) is a subsequence of \( s_2 \)).

Remark 4.6. Someone should present the part of Oriented Flip Graphs & Noncrossing Tree Partitions about the shard intersection order of \( \Psi(L) \). They should explain the CU-labeling of \( \Psi(L) \) that we construct and how it is intrinsic to \( \text{FG}(T) \).

Remark 4.7. Someone should present Petersen’s On the shard intersection order of a Coxeter group paper (using some basic definitions from Reading’s Noncrossing partitions the shard intersection order).

Definition 4.8 (Reading). Let \( L \) be a congruence-uniform lattice with CU-labeling \( \lambda : \text{Cov}(L) \to P \). Let \( x \in L \) and let \( y_1, \ldots, y_k \) be the elements of \( L \) satisfying \((y_i, x) \in \text{Cov}(L) \). Define the shard intersection order of \( \Psi(L) \) to be the collection of sets of the form
\[
\psi(x) := \{ \text{labels appearing between } \bigwedge_{i=1}^k y_i \text{ and } x \} = \{ \lambda(w, z) : \bigwedge_{i=1}^k y_i \leq w < z \leq x, (w, z) \in \text{Cov}(L) \}
\]
partially ordered by inclusion.

Problem 4.9. Describe the shard intersection order of \( \text{Bic}(T) \).

a) Construct a CU-labeling \( \lambda : \text{Cov}(\text{Bic}(T)) \to S \) where \( S \) is variation of the poset Seg(T).

b) Is \( \Psi(\text{Bic}(T)) \) a lattice?

b) Is it EL-shellable?