Let's start with triangulations of an $n$-gon.
This gives a bijection between triangulations of \((n+2)\)-gons and binary trees with \(n\) nodes.
Now add all possible children ...

... so they are also in bijection with complete binary trees with $n+1$ leaves.
How to count them?
Partition the leaves in the complete binary tree according to the left- and right-subtrees below the root.

![Diagram of a complete binary tree with leaves partitioned into two subtrees.]

complete binary tree with \( k \) leaves

some with \( n+1-k \) leaves

for some \( k=1,2,\ldots,n \)

Hence \( c_n = \# \text{ complete binary trees with } n+1 \text{ leaves} \)

satisfies \( c_n = \sum_{k=0}^{n-1} c_k c_{n-k-1} \)

(note re-indexing)
Base case: \( \text{C}_0 = 1 \)

\[
\begin{array}{c|c|c|c|c|c|c|}
 n & 0 & 1 & 2 & 3 & 4 & 5 \\
\hline
\text{C}_n & 1 & 1 & 2 & 5 & 14 & 42 \\
\end{array}
\]

\[ \text{C}_n = \text{the } n^{th} \text{ Catalan number} \]

\[ n + 1 \binom{2n}{n} \]

Not obvious; well-known. See e.g. Wikipedia

There are over 200 objects counted by Catalan numbers; see Stanley's book "Catalan Numbers"

E.g. Dyck paths
DEF'N: A Dyck path is a walk from \((0,0)\) to \((n,n)\) in \(\mathbb{Z}^2\) taking unit steps up (U) and right (R), staying weakly above \(y=x\).

e.g. \(n=3\)

- URURUR
- URUURR
- UURRUR
- UURURR
- UUURRR
DEF: A rooted planar tree is a rooted tree in which each vertex has a linear ordering (left to right) of its children.

E.g. \[ \begin{array}{c} \text{\textbf{\textless}} \\
\text{\textgreater} \end{array} \neq \begin{array}{c} \text{\textbf{\textgreater}} \\
\text{\textless} \end{array} \]

They biject with Dyck words, via the planar code of the RPT.
You get the planar code by starting at the root and walking around the tree, recording:
- U when moving away from root,
- R when moving toward the root.

E.g.

```
URUUURUURURURRRURURRR
```
There is a bijection from binary trees to Dyck paths (omitted here).

REU Exercise 12

Describe a bijection

\[
\{2 \times n \text{ standard Young tableaux}\} \leftrightarrow \{321\text{-avoiding permutations in } S_n\}
\]

(a filling of \( n \choose 2 \))

with 1, 2, ..., 2n each appearing exactly once, increasing in rows and columns

\[
\begin{array}{c}
\text{e.g.} \\
\text{n=5} \\
12458 \\
347910
\end{array}
\]

\(\pi=\pi_1\pi_2\ldots\pi_n\)

having no \(i<j<k\) with \(\pi_k>\pi_j>\pi_i\)

\(\pi=24351\) fails

\(12345\) \(\{\) OK

\(13254\) \(\}\)
One more set of objects...

**DEF’N:** A rigged configuration is a partition \( \lambda \) together with integers \( J_i \) for each part \( \nu_i \) of \( \nu \), satisfying \( 0 \leq J_i \leq P_i \), where \( P_i \) is the \( i \)th vacancy number:

\[
2n - 2 \sum_{j} \min(\nu_j, \nu_i)
\]

The sum counts these boxes.
The vacancy numbers:

Example:

\[
\begin{array}{|c|c|}
\hline
0 & 1 \\
\hline
2 & 0 \\
\hline
\end{array}
\]

= 

\[
\begin{array}{|c|c|}
\hline
0 & 0 \\
\hline
0 & 1 \\
\hline
\end{array}
\]

The rigging:

- It associates a multiset of values to each part size.

Example:

\[
\begin{array}{|c|c|c|}
\hline
0 & 0 \\
\hline
8 & 5 & 5 \\
\hline
20 & 10 \\
\hline
34 & 5 & 5 & 1 \\
\hline
\end{array}
\]

May as well reorder weakly decreasing.
Multiplicity notation:
\[ V = 1^{m_1} 2^{m_2} 3^{m_3} \ldots \]
means \( V \) has \( m_3 \) parts of size 3
\[ P_e = 2n - 2 \sum_i m_i \min(i, l) \]
(= vacancy number for all parts \( 2i - l \))

There is a fermionic formula
\[ c_n = \sum_{V \uparrow \varnothing} \prod_{l=1}^{\infty} \left( \frac{m_l + p_l}{m_l} \right) \]
“\( V \) is a partition of \( n \)”
which, using the fact that
\[ \binom{a+b}{a} = \# \left\{ \text{multisets of size } a \text{ from } \{0, 1, \ldots, b\} \right\} \]
... suggests the following.

**CLAIM**: There is a bijection

\[
\begin{array}{c}
\text{RC} \\
\xrightarrow{\Psi} \\
\text{RPT}
\end{array}
\]

given by reading the RC from top to bottom, adding a path of length \(2i\) at the \(j_i\)-th possible position.

---

**E.g.**
(Sorry, note-taker got a little lost at this next stage ... )

The key issue is in the numbering of the possible positions (purple) when adding the group of next smallest parts.
Given the Dyck word, define at the $k$th stage of the bijection $\Phi$

$$P_l = k - 2 \sum_{i} \min(v_i, l)$$

and define $\Phi$ recursively by adding a box to the longest row of $v$ such that $J_i = P_i$ (which we call singular) and keep the row singular if we add $R$, and do nothing for $U$. 

\[ \Phi : D_n \rightarrow RC_n \]
e.g. 

\[
\begin{align*}
\emptyset & \xrightarrow{U} \emptyset \\
& \xrightarrow{R} \quad 0 \\
U & \xrightarrow{ } 1 \\
& \xrightarrow{ } 2 \\
R & \xrightarrow{ } 1 \\
& \xrightarrow{ } 2 \\
U & \xrightarrow{ } 3 \\
& \xrightarrow{ } 2 \\
R & \xrightarrow{ } 2 \\
& \xrightarrow{ } 1 \\
\end{align*}
\]
Can check

\[ \begin{array}{ccc}
0 & 2 & 0 \\
2 & 1 & 0 \\
\end{array} \quad \Psi \rightarrow \begin{array}{c}
\uparrow \\
\downarrow \mathrm{planar} \\
\mathrm{code} \\
\end{array} \]

\[
\text{URUURURRRUR}
\]

THM (Reynolds '15)
This diagram commutes

\[
\begin{array}{c}
\Phi \\
\end{array} \quad \Downarrow \mathrm{planar} \\
\mathrm{code} \quad \begin{array}{c}
\Phi \quad D \\
\end{array}
\]

\[
\begin{array}{c}
\Psi \\
\end{array} \quad \begin{array}{c}
\downarrow \mathrm{planar} \\
\mathrm{code} \\
\end{array} \quad \begin{array}{c}
\downarrow \mathrm{planar} \\
\mathrm{code} \\
\end{array}
\]

RC \rightarrow RPT
REU EXERCISE 13

Show $\Psi$, $\Phi$ are bijections
(directly, without using Reynolds's Thm)

Two statistics on Dyck paths

Area = # full boxes under the Dyck path

Bounce = the sum of the positions of the bounce path
Area = 8 (\# of boxes marked x)

Bounce = 1+2+5 = 8

The green bounce path bounces off the red Dyck path and off the diagonal y=x.
DEF. N: The \((q, t)\)-Catalan number

\[ C_n(q, t) := \sum_{d \in D_n} \frac{\text{area}(d) \cdot \text{bounce}(d)}{t} \]

THM (Garsia, Haglund et al)

\[ C_n(q, t) = C_n(t, q) \]

OPEN PROBLEM (Not REU!)

Prove this combinatorially.

Their proof is algebraic.
REU PROBLEM 5

Determine area and bounce on RC's under $\Phi$, i.e.
find statistics $\alpha, \beta$ such that

area $= \alpha \circ \Phi$
bounce $= \beta \circ \Phi$

REU EXERCISE 14

(i) Find a bijection
[complete binary trees] $\leftrightarrow$ RPT

(ii) Find the definition of area on RPT under planar code.