REU 2016 Day 8 V. Reiner

Faces of Gelfand-Tsetlin polytopes

(1) GT-patterns & polytopes
(2) Polytope review \( + \frac{2}{3} \) (REU Problem 8)
(3) Flag numbers & cd-index \( + \frac{1}{3} \) (REU Problem 8)

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(2) GT-patterns

Recall \( s_\lambda (x_1, \ldots, x_n) = \text{Schur function} = \sum_{\text{Semistandard tableaux } T \vdash \lambda} x^T \)

All this set \( \Rightarrow \) \( \text{entries in } \{1, 2, \cdots, n\} \)

\( \lambda = (1 \leq 1 \leq 3) \)

\[ s_\lambda (x_1, x_2, x_3) = x_1^3 x_2 x_3 + x_1^2 x_2^2 x_3 + x_1^2 x_2 x_3^2 + x_1 x_2^3 + x_1^3 + x_3^3 \]

\( \{111, 112, 113, 122, 123, 133\} \) = \( \text{SST}([3]) \)

\( \text{PROP(Gelfand-Tsetlin)} \) There is a bijection

\[ \text{SST}(\lambda; n) \quad \sim \quad \frac{\text{GT}(\lambda; n)}{z} = \text{GT-patterns with top row } \lambda \]

\[ \text{shape}(T) \]

\[ (x_i^{(n)} \text{ if } |T_i| = n) \]

\( \lambda = (0, 0, 2, 4, 4) \)

\( \eta = 5 \)
proof: Think about how far in rows 1, 2, ..., n the values extend versus how far in rows 1, 2, ..., n-1 the values \( s_{n-1} \) extend.

The latter distances interlace the former: \( \cdots \bullet \bullet \bullet \bullet \) (and it's reversible)

DEFN: The GT-polytope \( GT(\lambda) := GT_{\mathbb{R}}(\lambda) \) is the solution set in \( \mathbb{R}^{(2)} \) with coordinates \( (x_{i,j})_{1 \leq i \leq j \leq n-1} \) to the same inequalities

EXAMPLES:

1. \( GT(\begin{array}{c} 1 \ 1 \ 1 \\ 0 \ 1 \ 1 \end{array}) \)

2. \( GT(\begin{array}{c} 1 \ 1 \ 1 \\ 0 \ 2 \ 1 \end{array}) \)

affinely isomorphic (defined below)
(2) In fact both $GT\left(\frac{3 \cdot 4 \cdot 5}{3! \cdot 4! \cdot 5!}\right)$ and $GT\left(\frac{3 \cdot 4 \cdot 5}{3! \cdot 4! \cdot 5!}\right)$ are still affinely isomorphic, but with some dilation involved.

(3) Given $\Lambda = (\Lambda_1, \Lambda_2, \Lambda_3)$, let's try to draw $GT(\Lambda) \subset \mathbb{R}^3$ coordinates $(x,y,z)$ starting with $\Lambda_1 \leq x \leq \Lambda_2 \leq y \leq \Lambda_3$.

Then impose $x \leq z$ and $z \leq y$, slicing off two wedges:

For this into SageMathCell:

```plaintext
Polyhedron(iegs = [ [0,-1,0,1], [0,0,1,-1], [1,1,0,0], [2,0,1,0], [2,-1,0,0], [3,0,-1,0] ], plot())
```

expresses the inequality $3+0\cdot x+(-1) y+0\cdot z \geq 20$ i.e. $y \leq 3$.
Using multiplicity notation: $\lambda = \lambda_1 \leq \cdots \leq \lambda_l = (\lambda_1 \leq \cdots \leq \lambda_l)$

(a) show that $\dim \text{GT}(\lambda) = \binom{\lambda_1}{2} - \sum_{i=1}^{l-1} \binom{\lambda_i}{2}$

(b) show that $\text{GT}(\mathbf{1}_{l-1})$ is an $(l-1)$-dimensional simplex, that is, the convex hull of $l$ points and $(l-1)$-dimensional.

e.g. $\text{GT}(\mathbf{1}) = \cdots$  
$\text{GT}(\mathbf{1}_1) = \cdots$
$\text{GT}(\mathbf{1}_2) = \cdots$

(c) Find an affine isomorphism $\text{GT}(\lambda) \sim \text{GT}(\lambda')$ for some linear map and automorphism of affine $Z_2$-symmetry of $\text{GT}(\lambda)$ and show that it gives a nontrivial

e.g. it should give $(18)(25)(37)(6)$ on $Z_2(1234)$
The REU problem will focus on facial structure of $\Gamma(n)$, so recall what faces are...

**DEFINITION:** A polyhedron $P \subseteq \mathbb{R}^d$ is a finite intersection $P = \bigcap_{i=1}^t H_i^+$

where each $H_i^+$ is a half-space $\{x \in \mathbb{R}^d : a_i x_1 + \ldots + a_i x_d \geq b_i\}$

with (affine) hyperplane $H_i = \{x \in \mathbb{R}^d : a_i x_1 + \ldots + a_i x_d = b_i\}$

E.g. $P_1$

A **face** of a polyhedron $P$ is an intersection $F = H \cap P$

where $H$ is the hyperplane for some half-space $H^+ \supseteq P$ called a supporting half-space

E.g. $F = H \cap P$ $= H_3 \cap P$

- vertices = 0-diml faces
- edges = 1-diml faces

$P$ itself is considered a face of $P$

The face poset $\mathcal{F}(P) = \{\text{faces of } P\}$ ordered under inclusion turns out to be a graded lattice of rank $\dim P + 1$

(see e.g. Ziegler's "Lectures on Polytopes")

E.g. $\mathcal{F}(P_1)$

Say $P_1, P_2$ are combinatorially isomorphic if $\mathcal{F}(P_1) \approx \mathcal{F}(P_2)$

Clearly $P_1, P_2$ affinely isomorphic implies this.
THM (T. McAllister) 2006

The map sending a $\Gamma(\lambda)$-pattern $\pi$ to the decomposition

$$\pi = \bigcup_{i=1}^{m} \Gamma_{i},$$

where the $\Gamma_{i}$ are the connected components of the "equal patches" in $\pi$

gives a poset isomorphism

$$F(\Gamma(\lambda)) \sim \{\text{GT(\lambda)-things}\text{ under refinement}\}$$

$$\text{decompositions } \bigcup_{i=1}^{m} \Gamma_{i} \text{ where}$$

- $\Gamma_{i}$ are connected
- $\Gamma_{i}$ are convex in this sense:
- restricted to the top row, $\bigcup_{i=1}^{m} \Gamma_{i}$ partitions like the parts of $\lambda$

E.g. $\lambda=111588$

\[
\begin{align*}
111588 & \quad \rightarrow \quad 6 \text{ tiles total}
\end{align*}
\]

REU PROBLEM 8(a): What is the diameter of $\Gamma(\lambda)$?

They seem remarkably small.

\[
\text{max distance in the 1-skeleton (vertices + edges)}
\]

E.g., diameter $\Gamma(1,2,3)=2$

CONJECTURE: $\text{diam } \Gamma(\lambda_{1}\leq\cdots\leq\lambda_{n}) \leq 2(n-2)$ for $n \geq 3$

with equality here $\iff \lambda_{1}<\lambda_{2}<\cdots<\lambda_{n}$

CONJECTURE: $\text{diam } \Gamma(0^{k},1^{k}) = 2$

CONJECTURE: $\text{diam } \Gamma(0^{1},1^{2}) = 2$

RMK: $\Gamma(\lambda)$ can have non-integral vertices for $n \geq 5$

$\lambda_{5}\ldots\lambda_{n}$

(Pelecova-McAllister) 2006
REU PROBLEM 8(b): Is there a hidden symmetry in $GZ(\lambda)$?

**Conjecture:** There is a non-trivial $Z/2Z$-combinatorial symmetry on $GZ(\lambda) \forall \lambda$

$\left( ? \right)$ **Conjecture:** $GZ(\lambda, \cdots, \lambda_n)$ has $Z/2Z \times Z/2Z$ symmetry

**Conjecture:** $GZ(0^1, 1^m, 2^l)$ has a hamnogenous symmetry group, that grows with $n$.

### 3. Flag numbers & cd-index

For a $d$-dimensional polytope $P$, the flag f-vector $(f_S : S \subseteq \{1, 2, \ldots, d\})$ records each flag number $f_S := \# \text{flags of faces in } P \text{ passing through}$

- $\text{ranks } S \text{ in the face poset}$
- but is highly redundant, and has values unnecessarily large...

**Example:** $P = GZ(\lambda, \lambda_2, \lambda_3)$ has

<table>
<thead>
<tr>
<th>$S$</th>
<th>$f_S$</th>
<th>$h_S$</th>
<th>$ab$-monomial</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\emptyset$</td>
<td>1</td>
<td>1</td>
<td>1baa</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
<td>6bba</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>11</td>
<td>10bba</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>5</td>
<td>5aba</td>
</tr>
<tr>
<td>12</td>
<td>22</td>
<td>5</td>
<td>5bba</td>
</tr>
<tr>
<td>13</td>
<td>22</td>
<td>10bba</td>
<td>10bba</td>
</tr>
<tr>
<td>23</td>
<td>22</td>
<td>6</td>
<td>6abb</td>
</tr>
<tr>
<td>123</td>
<td>44</td>
<td>1</td>
<td>1bbb</td>
</tr>
</tbody>
</table>

Better is the flag h-vector $(h_S : S \subseteq \{1, 2, \ldots, d\})$

defined via

$$h_S := \sum_{T \subseteq S} (-1)^{\lvert S \setminus T \rvert} f_T$$

(or equivalently via inclusion-exclusion

$$f_S = \sum_{T \subseteq S} h_T$$

**FACTS:**

- $h_S \geq 0$
- $h_S = h_{S \cup \{d\}}$
- $h_{S \cup \{d\}} = h_S$

Even better is the cd-index of $P$

*defined by 1st creating the ab-polynomial

$$\sum_{S} h_S \, \text{ab}(S)$$

*where $\text{ab}(S)$ is a noncommutative degree monomial $a(babba \cdots)$

and then writing it in terms of $a = ab$ and $b = ab + ba$. It can always be done! (Bayer-Billera-Rieke)
REU PROBLEM 8(c):

Compute the cd-index for some families of \( GT(n) \),

- \( GT(0^k 1^{n-k}) \)
- \( GT(0^1 1^{n-2}) \)
- \( GT(\lambda_1 < \lambda_2 < \ldots < \lambda_n) \), i.e., \( GT(1^2 3^4 \ldots n^1) \)

RMK: SAGE has the face poset \( F(P) \) and its flag f-vector flag-vector but not cd-index (that I could find).

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REU EXERCISE 20

(a) Show that the ring map defined by

\[
\begin{align*}
\mathbb{Z}[c, d] & \quad \rightarrow \mathbb{Z}[a, b] \\
c & \rightarrow a+b \\
d & \rightarrow ab + ba
\end{align*}
\]

is injective, but not surjective. In particular, cd-indices are unique.

(b) Show that as a subset of \( \mathbb{R}^d \), the set \( \{ \text{flag f-vectors (f_3)} \} \) affinely spans a space of dimension at most \( F_d - 1 \) where \( \{ F_0 = F_1 = 1 \}
\)

\( F_n = F_{n-1} + F_{n-2} \)

are Fibonacci numbers.

It turns out that equality holds (Bayer–Billera).