Shards and noncrossing tree partitions

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4. The structure of noncrossing tree partitions
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Fix a tree $T$ embedded in a disk with exactly its leaves on the boundary and whose interior vertices (the vertices not on the boundary) have degree at least 3.
We obtain the following diagram of posets defined from $T$:

\[
\begin{array}{c}
\text{Bic}(T) \xrightarrow{\psi} \Psi(\text{Bic}(T)) \\
\downarrow \phi \quad \downarrow ? \\
\overrightarrow{FG}(T) \xrightarrow{\psi} \Psi(\overrightarrow{FG}(T)) \sim \text{NCP}(T)
\end{array}
\]

Goal: Understand the combinatorics of NCP($T$)
The poset $\text{NCP}(T)$ is called the *noncrossing tree partitions* of $T$. In this part of the talk, we will discuss our research of the following properties of $\text{NCP}(T)$:

1. $\text{NCP}(T)$ is a lattice
2. $\text{NCP}(T)$ is graded (conjecture)
3. $\text{NCP}(T)$ is not self-dual
4. How to count the maximal chains in $\text{NCP}(T)$
What is NCP(T)?

For a tree $T$, a segment $s = (v_0, \ldots, v_t) = [v_0, v_t]$ with $t \geq 1$ is a sequence of interior vertices of $T$ that takes a “sharp” turn at each $v_i$. In particular, the interior vertices of $T$ are not segments.

Example

In the tree below, $(1, 5)$ and $(2, 4, 6)$ are segments. The sequence $(1, 3)$ is not a segment.
A noncrossing partition \( \mathbf{B} = (B_1, \ldots, B_k) \) is a set partition of the interior vertices of \( T \) where

- the vertices in \( B_i \) can be connected by *red admissible curves* (i.e. curves whose endpoints define segments of \( T \) and leave their endpoints to the right), where any pair of such curves can only agree at their endpoints, and
- red admissible curves connecting vertices of \( B_i \) do not cross those of \( B_j \) for \( i \neq j \).

We let \( \text{NCP}(T) \) denote the poset of noncrossing tree partitions ordered by refinement.

**Example**

\( \mathbf{B} = \{\{1, 4, 6\}, \{2, 3\}, \{5\}\} \) is an element of \( \text{NCP}(T) \).
Theorem (Garver-McConville)

The poset $\text{NCP}(T)$ is a lattice.
Before we talk about the structural properties of $\text{NCP}(T)$, we need to discuss the relevant lattice theory.

**Definition**

A lattice is called *congruence-uniform* if it can be constructed from a single point using interval doublings.

Here is an example of a lattice constructed from interval doublings:
A lattice is congruence-uniform if and only if it admits an edge labeling known as a \textit{CU-labeling}.

In fact, the colors on the edges of the picture above form a CU-labeling, where the color set is ordered $s \leq t$ if the color $s$ appears before $t$ in the sequence of doublings.
$L$ a lattice
\lambda a CU-labeling of $L$
$x \in L$

$\Psi(L)$ Shard intersection order

$\Psi(L)$ consists of sets

$$\psi(x) = \{ \text{labels appearing between } \bigwedge_{i=1}^{k} y_i \text{ and } x \}$$

where $\{y_i\}_{i=1}^{k}$ is the set of elements immediately below $x$ in $L$. The partial ordering on $\Psi(L)$ is inclusion. We call the interval $[\bigwedge_{i=1}^{k} y_i, x]$ the facial interval corresponding to $x$. 
Theorem (Garver-McConville)

For a tree $T$, $\text{NCP}(T)$ is isomorphic to $\Psi(\overrightarrow{FG}(T))$.

This brings us to one of the main objects in our project:

Conjecture

The lattice $\text{NCP}(T)$ is graded by the number of blocks in a partition.
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How we want to prove this conjecture:

- Show that every covering relation in $\text{NCP}(T)$ is given by merging two blocks of a partition (which is what happens with $\text{NC}(n)$).
- To do this, it suffices to show that if we can merge $m$ blocks of $B$, $m \geq 3$, then we can merge $m - 1$ blocks.
- To show the above, we work with $\overrightarrow{FG}(T)$. We know that $B$ corresponds to a facial interval in $\overrightarrow{FG}(T)$. We want to show that it is contained in a facial interval “one dimension lower” than the entire lattice.

$$B \mapsto \psi(x) \sim [a, x] \subsetneq [a', x'] \subsetneq \overrightarrow{FG}(T)$$
Garver and McConville defined a bijection $\text{NCP}(T)$ called the \textit{Kreweras Complement}. The Kreweras complement sends a partition with $m$ blocks to a partition with $\#V^c(T) + 1 - m$ blocks. A corollary of this map and the previous conjecture is the following:

\begin{corollary}
The lattice $\text{NCP}(T)$ is rank-symmetric.
\end{corollary}

The above property is shared by $\text{NC}(n)$. A natural question to ask is: How many of the nice properties of $\text{NC}(n)$ carry over to $\text{NCP}(T)$? We provide a partial answer here:

\begin{theorem}
In general, $\text{NCP}(T)$ is not self-dual.
\end{theorem}
We conclude our discussion of NCP($T$) with a method of calculating the number of maximal chains, denoted $mc(T)$.

We will exploit the following fact in order to obtain recursions: let $\{a_i\}_{i=1}^n$ be the set of coatoms of NCP($T$); then

$$mc(T) = \sum_{i=1}^n mc([\hat{0}, a_i]).$$

From here, we can note that $[\hat{0}, a_i]$ is isomorphic to the product of two noncrossing tree partitions of smaller trees, as shown by the following picture:
We have that $[\hat{0}, a_i] \cong \text{NCP}(T_1) \times \text{NCP}(T_2)$, where
Using this method, we can count the maximal chains of the $k$th star-graph, denoted $S_k$, which is the family of trees of the following form:

$$S_5$$

$k = \# \text{ of edges attached to central vertex}$

We get that $mc(S_k) = \frac{k!F_{k+1}}{2}$, where $F_{k+1}$ is the $(k + 1)$th fibonacci number.
Segments $S_1, S_2 \in \text{Seg}(T)$ whose composition is also in $\text{Seg}(T)$ are composable.
A subset $B \subset Seg(T)$ is **closed** if for any composable $S_1, S_2 \in B$, we have $S_1 S_2 \in B$. 
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$Bic(T)$ is a poset whose elements are biclosed sets $B \subset Seg(T)$, partially ordered by inclusion.
We will explicitly demonstrate the CU-labeling for $Bic(T)$. 
For a segment $[a, c]$ with vertex $b$ in between, we say that $[a, b]$ and $[b, c]$ constitute a break of $[a, c]$
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Each of \([a, b]\) and \([b, c]\) is a **split** of \([a, c]\) corresponding to that break.
Recall that a CU-labeling is a map $\lambda: \{\text{covering relations of } Bic(T)\} \to P$ for some poset $P$ of labels.
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We choose $P$ with elements of the form $S_\Delta$ where $S \in Seg(T)$ and $\Delta$ is a set of splits of $S$. The partial ordering is given by $S_\Delta \succeq Q_\mu$ if $S$ contains $Q$. 
Covering relations in $Bic(T)$ look like:

\[ B \cup \{s\} \]
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\[
B \cup \{s\} \\
| \\
S_\Delta \\
B
\]
Example:
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Covering relations look like

\{2, 12, 23, 123\}

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{2, 12, 23, 123}

123_Δ

{2, 12, 23}
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\[ \{2, 12, 23, 123\} \]

\[ 123_{\{23,\}} \]

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\{2, 12, 23\}

\{23, 12\}
\( \Psi(\text{Bic}(T)) \) has a maximum element.

If we can show that for all \( C, D \in \text{Bic}(T) \), there exists some \( B \in \text{Bic}(T) \) such that \( \psi(C) \cap \psi(D) = \psi(B) \), then we can conclude that \( \Psi(\text{Bic}(T)) \) is a lattice.
Elements of $\psi(B)$ are those of the form $S_\Delta$ where $S$ is a composition of some of $S_1, S_2, \ldots, S_m$ and $\Delta$ is a set of splits of $S$ with certain stipulations.
Vertices within $S$ which are endpoints of some $S_i$ correspond to faultline breaks.

Other vertices correspond to non-faultline breaks.
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![Diagram showing the splits $S_{i1}, S_{i2}, S_{i3}$]
What splits of $S$ are in $\Delta$?

For non-faultline breaks, these are predetermined by $\Delta_1, \Delta_2, \ldots, \Delta_m$. For each faultline break, there is an independent choice.
Labels in $\psi(C) \cap \psi(D)$ are of the form $S_\Delta$ where:

\[ S_\Delta \text{ must simultaneously be a composition of } S_i \text{'s and } Q_i \text{'s.} \]

Furthermore, the splits determined by the corresponding $\Delta_i$'s and $\mu_i$'s must be compatible.
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$S$ must simultaneously be a composition of $S_i$’s and $Q_j$’s:
The only breaks for which there is a choice of what split of $S$ to include in $\Delta$ are when the break is a faultline for $S$ viewed as a composition of $S_i$’s and $S$ viewed as a composition of $Q_i$’s.
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Call an element of $\psi(C) \cap \psi(D)$ \textbf{pseudominimal} if it does not contain any double faultlines in its composition pair.
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1. $S$ is a composition of the segment parts of pseudominimal labels.
2. The only choices for which splits of $S$ to include in $\Delta$ occur at breaks where two such pseudominimal labels are joined together.
Pseudominimal elements of $\psi(C) \cap \psi(D)$ generate $\psi(C) \cap \psi(D)$ the same way $\psi(B)$ is generated by the labels on its covering relations.

We can conceivably take

$B = \vee \{ \text{pseudominimal elements of } \psi(C) \cap \psi(D) \}$ to obtain

$\psi(B) = \psi(C) \cap \psi(D)$. 
For the star graph $S_k$:

$$|NCP(S_k)| =$$

![Diagram of a star graph $S_k$ with a central node connected to five peripheral nodes.](image)
For the star graph $S_k$:

$$|NCP(S_k)| = 2|NCP(S_{k-1})| + |NCP(S_{k-2})|$$
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with $|NCP(S_3)| = 14, |NCP(S_4)| = 34$
Straight trees like

\[
\begin{array}{c}
\text{\includegraphics[width=0.3\textwidth]{tree.png}}
\end{array}
\]

are analogous to classical non-crossing partitions.
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