1. $q$-count

$$\left[^n \ell \atop k \ell \right]_q := \frac{[n]_q! [\ell]_q!}{[n-k]_q! [\ell-k]_q!},$$

$q$-binomial coefficient.

where $[n]_q := (q^1 - q^{-1})(q^2 - q^{-2}) \cdots (q^n - q^{-n})$ and $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = \frac{1 - q^n}{1 - q}$

\[ \left[^n \ell \atop k \ell \right]_q \]

Example:

$$\left[^4 \atop 2 \right]_q = \frac{[4]_q! [3]_q! [2]_q! [1]_q!}{[2]_q! [1]_q! [1]_q! [1]_q!} = \frac{[4]_q [3]_q [2]_q [1]_q}{[2]_q [1]_q [1]_q [1]_q} = \frac{(1 + q + q^2)(1 + q + q^2)}{1 + q} = (1 + q^2)(1 + q + q^2) = 1 + q + 2q^2 + q^3 + q^4 \quad \frac{3}{2} \cdot 1 - \frac{1}{q} = 6 = (q) \checkmark$$

**REV EXERCISE 3:**

(a) Show $\left[^n \ell \atop k \ell \right]_q = \sum_{\substack{\text{partitions } \lambda \\text{ with } \lambda \leq n-k \\text{ and at most } k \text{ parts} \\text{i.e. } \lambda \lessgtr \underbrace{1 \ldots 1}_{n-k} \\text{ such that } \sum \lambda_i = \ell}} q^{\lambda_1}$ (so $\left[^n \ell \atop k \ell \right]_q \in \mathbb{N}[q]$)

(b) Show $\left[^n \ell \atop k \ell \right]_q = \left[^{n-1} \ell \atop k-1 \ell \right]_q + q \left[^{n-1} \ell \atop k \ell \right]_q$

(c) Show that, setting $q = p^d$ a prime power, $\left[^n \ell \atop k \ell \right]_q = \# \{ k \text{-dimensional } \mathbb{F}_q \text{-linear subspaces } X \text{ of } \mathbb{F}_q^n \}$
2. Cyclic actions

Evaluating \([k_n]^q\) at \(q\)th powers of \(x_n^q\) gives \(\zeta\)th roots of unity in \(C\),

turns out to count something too, related to an \(n\)-cycle \(c = (1, 2, \ldots, n)\)

permuting \(k\)-element subsets of \(\{1, 2, \ldots, n\}\)

**Example:** \(n = 4\)

\(c = (1, 2, 3, 4)\) permutes

\[\{1, 2, 3, 4\} \rightarrow \{2, 3, 4, 1\}\]

\[\{1, 2, 3\} \rightarrow \{2, 3, 1\}\]

**Theorem:** (R. Stanley-White, 2001)

When \(c = (1, 2, \ldots, n)\) permutes \(k\)-element subsets of \(\{1, 2, \ldots, n\}\),

\([k_n]^q\) counts the number fixed by \(c^q\).

**Example:**

\[x_2^q = 1 + q + 2q^2 + q^3 + q^4\]

\(x_4 = i = e^{\frac{2\pi i}{4}}\)

\(x_i^q = i^q = i^n\) since no 2-subsets \(\{i, j\}\) are fixed by \(c\).
(3) REV EXERCISE 4: Prove the previous THM by this strategy...

(a) Show that if \( f \) is any primitive \( m \times n \) root-of-unity, then

\[
\begin{pmatrix} m' \\ k' \end{pmatrix}_{q^m \circ f} = \left( \begin{pmatrix} m' \\ k' \end{pmatrix} \right) \begin{pmatrix} m'' \\ k'' \end{pmatrix}_{q^m \circ f}
\]

where \( n = m/n' + n'' \), \( 0 \leq n' \leq m - 1 \)

\[ l = n' k' + l'' \], \( 0 \leq l'' \leq m - 1 \)

i.e.

\[
\begin{pmatrix} n'' \\ k'' \end{pmatrix} = \begin{pmatrix} m - n' \\ l'' \end{pmatrix} \frac{l''}{m'} \frac{m'}{m - n'}
\]

(b) Apply this when \( f = f_3^d \), and compare to brute force count of \( k \)-subsets fixed by \( c_3^d \)

---

3. *q*-Catalan

The Catalan number \( C_n = \frac{1}{n+1} \binom{2n}{n} \) counts

- triangulations of an \((n+2)\)-gon
- noncrossing set partitions of \( \{1, 2, \ldots, n\} \)

**Example:** \( C_4 = \frac{1}{5+1} \binom{2\cdot4}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{5 \cdot 4 \cdot 3 \cdot 2} = 14 \)

Counts both...

<table>
<thead>
<tr>
<th>6-gen-triangulations</th>
<th>noncrossing partitions of ( {1, 2, 3, 4} )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 + 4 + 2 + 4 + 2 + 1 = 14</td>
</tr>
</tbody>
</table>

THM (RSW) 2004: MacMahon's *q*-Catalan number \( C_n(q) = \frac{1}{2n+1} \binom{2n}{n} q^d \) has

(a) \( C_n(q) \bigg|_{q = f_3^d} \) counting \((n+2)\)-gon triangulations fixed by \( c_{n+2}^d \)

(b) \( C_n(q) \bigg|_{q = f_3^d} \) counting noncrossing partitions of \( \{1, 2, \ldots, n\} \) fixed by \( c_{n}^d \)
4. Dihedral action

In fact, \( \mathbb{Z}_n \) isn't just cycled by \( c = (1, 2, \ldots, n) \) and powers \( c^i \),

it carries an action of the dihedral group \( I_2(n) \) of order \( 2n \)

: symmetries of a regular \( n \)-gon

**Examples:**

\( n = 4 \)

\( n = 5 \)

Abstractly, \( I_2(n) = \langle s, t \rangle \),

\( s^2 = t^2 = e = (st)^n \)

\( = \langle s, c \rangle \)

\( s^2 = c^n = e \),

\( scs = c^{-1} \)

\( \{ e, sc, sc^2, \ldots, sc^{n-1} \} \),

\( n \) rotations

\( \{ e, c, c^2, \ldots, c^{n-1} \} \),

\( n \) reflections
The symmetries picture gives a representation $I_2(m) \to \text{GL}_2(\mathbb{R})$ into some $\text{GL}_n(\mathbb{R})$.

$\begin{align*}
I_2(m) & \overset{\rho}{\to} \text{GL}_2(\mathbb{R}) \\
(c & \to \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \quad \text{where} \quad \theta = \frac{2\pi}{n} \\
s & \to \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{if we make } s \text{ swap } x,y \text{ axes}
\end{align*}$

The permutation actions on vertices \{1, 2, 3, \ldots, n\}, or $k$-subsets of \{1, 2, \ldots, n\}, or triangulations or non-crossing partitions give various other representations $I_2(m) \to \text{GL}_n(\mathbb{R})$.

Sending group elements to permutation matrices is direct sums of irreducible representations.

**GENERAL PROBLEM**: Decompose these other representations into irreducible representations, that is, those with no non-zero subspaces stabilized (even working over $\mathbb{C}$, instead).

\[ \text{THM (Maschke)} \quad \text{This can always be done for finite reps } G \to \text{GL}_n(\mathbb{C}) \]

\[ \text{THM (Frobenius)} \quad \text{One can do it by computing the character } \chi_\rho : G \to \mathbb{C} \quad \rho \to \text{Trace} \]

and decomposing it uniquely as a sum of the irreducible rep's characters.

There are various tricks (orthogonality relations) that make it easier.

**EXAMPLE**: $G = I_2(4) = \{e, c, c^2, c^3, s, sc, sc^2, sc^3\}$

has 5 conjugacy classes

$\begin{align*}
e & | c, c^3 | c^2 | s, sc | sc, sc^3
\end{align*}$

and 5 irreducible reps & characters:

\[ sc^2, sc^3 \]
### Character Table

<table>
<thead>
<tr>
<th></th>
<th>e</th>
<th>c, c²</th>
<th>c²</th>
<th>s, s², c²</th>
<th>s², s², c²</th>
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<tr>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
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<td>-1</td>
<td>1</td>
<td>-1</td>
<td>+1</td>
</tr>
<tr>
<td>Xₜ</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>+1</td>
<td>-1</td>
</tr>
</tbody>
</table>

\[\chi_{X_t} = \chi_{X_{def}} \] 
\[\chi_{X_{def}} \]

<table>
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<tr>
<th></th>
<th>4</th>
<th>0</th>
<th>0</th>
<th>2</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>X₁₂₃₄</td>
<td>6</td>
<td>0</td>
<td>2</td>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

\[\chi_{X_{(2-subsets)}} = X_{(1,2,3,4)} + X_{(1,2)} + X_{(3,4)} \] 
\[\chi_{X_{(2-subsets)}} = X_{(1,2,3,4)} + X_{(1,2)} + X_{(3,4)} + X_{(def)} \]

Can expand \[\chi_{X_{(1,2,3,4)}} = 1 + Xₜ + X_{def} \] uniquely \[\chi_{X_{(2-subsets)}} = 1 + Xₜ + X₅ + X_{def} \]

---

### REU Problem 2

Expand into irreducibles:

(a) \(I_2(n)\) permuting \(\{k\text{-subsets of } \{1,2,\ldots,n\}\}\)
(b) \(-1\) \(\{\text{non-crossing partitions of } \{1,2,\ldots,n\}\}\)
(c) \(I_2(m,n)\) \(-1\) \(\{\text{triangulations of } (m,n)\text{-gon}\}\)

- What binomial Catalan identities ensue?
- Does \(C_n(q)\) help describe the characters at all?
- What about Garsia & Haiman's (q,t)-Catalan number \(C_n(q,t)\)?