The rule of three

Let $u_1, u_2, \ldots, u_n$ be non-commuting variables.

\[ e_3 = u_1 u_2 u_3 \neq u_2 u_1 u_3 = \ldots \]

Let $e_k = \sum_{i_1 < i_2 < \ldots < i_k} u_{i_1} u_{i_2} \ldots u_{i_k}$

**EXAMPLE**

$u_1 = a$, $u_2 = b$, $u_3 = c$

$e_1 = a + b + c$

$e_2 = ca + cb + ba$

$e_3 = cba$

\[
e_k(u_S) = \sum_{i_1 < i_2 < \ldots < i_k} u_{i_1} \ldots u_{i_k}
\]

**EXAMPLE** $e_2(u_{\{1,3\}}) = ca$
THEOREM (Kirillov, Blasiak-Fomin)
The following are equivalent:

- For any $k,l$ and $S \subseteq \{1,2,...,N\}$
  $$e_k(u_S)e_l(u_S) = e_l(u_S)e_k(u_S)$$

- These special cases hold:
  $$e_1(u_S)e_2(u_S) = e_2(u_S)e_1(u_S) \forall S \text{ with } 2 \leq |S| \leq 3$$
  $$e_1(u_S)e_3(u_S) = e_3(u_S)e_1(u_S) \forall S \text{ with } |S| = 3$$

EXAMPLE

$N=3$, $a,b,c$

$$ba(a+b) = (a+b)ba$$

$$\iff [bba+bab = aba+bba]$$

$$\Rightarrow (ca+ba+cb)(a+b+c) = (a+b+c)(ca+ba+cb)$$

$$ca+ca+ac \quad ca+ab+ac \quad ca+ab+ac \quad ca+ab+ac$$

$$+bca+bab+bac = +bca+bab+bca$$

$$+cba+cbb+cba$$

$$\iff [cab+bac = bca+abc]$$
\[(a+b+c)\,cba = cba(a+b+c)\]

\[acba + bdcba + c\,cba = cbab + cbac\]

\[acba + (bda+cda) = cbac + c(baa+bab)\]

\[\Rightarrow \boxed{acba + cbca = cbac + caba}\]

**REU EXERCISE 12** Prove these imply
\[(ba+ca+cb)\,cba = cba(ba+ca+cb)\]
3 more settings where such a "rule of three" might exist... (or a "rule of four"... ?)

**Quasisymmetric functions**

\( \alpha \) a composition
\( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l) \), \( \alpha_l \in \mathbb{Z}_{>0} \)

**EXAMPLE** \( \alpha = (2, 3, 1) \)

Monomial quasisymmetric fn
\( M_\alpha = \sum \chi_{i_1}^{\alpha_1} \chi_{i_2}^{\alpha_2} \cdots \chi_{i_l}^{\alpha_l} \) \( i_1 < \ldots < i_l \)

**EXAMPLE**
\( M_{(4,1)} = e_2 = \chi_1^1 \chi_2^1 + \chi_1 \chi_3^1 + \cdots \)
\( M_{(2,1)} = \chi_1^2 \chi_2^1 + \chi_1 \chi_3^1 + \chi_2^2 \chi_3^1 + \cdots \)
REU PROBLEM 5(a)
Is it true that there is a finite list of initial commutations that makes all of the $M_a$'s commute?

**Loop symmetric functions (T. Lam)**

Variables come in flavors:
- $a, b, c, A, B, C$
- $e_1 = a + b + c$
- $E_1 = A + B + C$
- $e_2 = bA + cA + cB$
- $E_2 = Ba + Ca + Cb$
- $e_3 = cBa$
- $E_3 = CbA$
REU PROBLEM 5(b)

Is it true that there is a finite list of initial commutations that makes all of the $e_k, E_k$ commute?

Schur Q-functions

Shifted Young diagrams $\longleftrightarrow$ partitions $\lambda$ with distinct parts

\[
\begin{array}{ccc}
1 & 2 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0
\end{array}
\longleftrightarrow \lambda = (4, 2, 1)
\]

Work in an alphabet $1 < 1' < 2 < 2' < \ldots$

A shifted tableau of shape $\lambda$ has

1) labels weakly increasing in rows and columns
2) each row has at most one $k$
3) each column has at most one $k'$

\[
Q_\lambda := \sum' x^T 
\]

shifted tableaux $T$ of shape $\lambda$
EXAMPLE

\[ T = \begin{array}{cccc}
1' & 1' & 3 & 4' \\
1 & 3 & 4
\end{array} \]

\[ x^T = x_3^2 x_2^2 x_4^2 \]

It turns out that the \( Q_n \) are symmetric functions, and they linearly span a subring \( \mathbb{Q}' \) of \( \Lambda \) which is generated by \( p_1, p_3, p_5, \ldots \) where \( p_k = x_1^k + x_2^k + \ldots \).
Let's suggest as analogues of $e_k$ here

$$Q, \ Q_{\square}, Q_{\square\square}, \ldots$$

$$Q_{\square} = 2(x_1 + x_2 + \ldots)$$

$$Q_{\square\square} = 2(x_1^2 + x_2^2 + \ldots) + 4(x_1x_2 + x_1x_3 + \ldots)$$

CONJ: $Q_{\square\square} \in \mathbb{Z}[Q, Q_{\square}, Q_{\square\square}, \ldots]$
REV PROBLEM 5(c)

Is it true that there is a finite list of
initial commutations that makes all
of the $\mathbb{Q}_{\text{odd}}$ commute?