Def: A minor of a matrix $M$ is the determinant of a matrix obtained from $M$ by selecting entries only in certain rows and columns.

**Ex.** $M = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 3 & 4 \end{bmatrix}$

$|M_{2,3,3,3}| = \begin{vmatrix} 4 & 1 \\ 2 & 4 \end{vmatrix} = 16 - 2 = 14 \quad \text{size 2}$

$|M_{2,1,3,2}| = \begin{vmatrix} 2 \end{vmatrix} = 2 \quad \text{size 1}$

**Def:** A **totally positive** (totally nonnegative) matrix is a matrix where all minors (of all sizes) are positive (nonneg.).

These are related to networks/planar graphs/wiring diagrams/cluster algs.

They have nice eigenvalues.

**Notation:** $[n] = 1, 2, \ldots, n$.

**Cauchy-Binet Formula:** Let $A, B$ be matrices of size $m \times n$, $n \times n$, $m \leq n$.

$$\det(AB) = \sum_{S \subseteq [m]} \det(A|S|) \det(B|S|)$$

where $A|S|, B|S|$ are $n \times n$

submatrices corresponding to $S$.

**Ex.** $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

$$n = 2, \quad m = 3$$

$$\det(AB) = \det(A|1,2|) \det(B|1,2|) + \det(A|1,3|) \det(B|1,3|) + \det(A|2,3|) \det(B|2,3|)$$

$$= \begin{vmatrix} 1 & 3 \end{vmatrix} \begin{vmatrix} 1 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 2 \end{vmatrix} \begin{vmatrix} 5 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 3 \end{vmatrix} \begin{vmatrix} 5 & 6 \end{vmatrix}$$

$$= (-2)(-2) + (-4)(-4) + (-2)(-2) = 4 + 16 + 4 = 24$$
If we want to know a minor of $XY$, we can think of $A, B$ from above as submatrices of $X, Y$.

**Picture:**

\[
\begin{array}{c}
\begin{bmatrix}
X
\end{bmatrix}
\begin{bmatrix}
Y
\end{bmatrix}
\end{array}
\begin{bmatrix}
A
\begin{bmatrix}
\begin{bmatrix}
B
\end{bmatrix}
\end{array}
\end{bmatrix}
\begin{bmatrix}
k
k
m
\end{bmatrix}
\end{array}
\begin{bmatrix}
= \begin{bmatrix}
k
k
m
\end{bmatrix}
\end{array}
\end{equation}

**Consequence:** Totally pos matrices, totally nonneg matrices are semirgs (sets with associative binary relation).

**LDU Factorization:** Let $X$ be a TNN nonsingular matrix. Then one can write $X = LDU$ where $L$ is a lower triangular matrix with 1's on the diagonal, $D$ is a (nonsingular) diagonal matrix, and $U$ is an upper triangular matrix with 1's on the diagonal. These matrices are given by:

\[
e_{ij} = \begin{vmatrix} X_{ij} \end{vmatrix} / \begin{vmatrix} X_{i,j} \end{vmatrix}, \quad u_{ij} = \begin{vmatrix} X_{ij} \end{vmatrix}, \quad d_{ii} = \begin{vmatrix} X_{i,i} \end{vmatrix}
\]

**Exercise:** The leading principal minors $|X_{[k,k]}|$ are positive for invertible TNN matrix $X$.

**Exercise:** The matrices in the LDU factorization are TNN.

**Def:** Let the Chevalley generators be $E_i(x) = \text{matrix with } 1's \text{ on the diagonal and } a \text{ in entry } i, i+1 \text{ and } E_i(x) = \text{matrix with } 1's \text{ on the diagonal and } a \text{ in entry } i, i+1$.

**Thm:** A TNN upper triangular matrix with 1's on the diagonal can be factored into $E_i(x)$'s with $x \geq 0$. A TNN lower triangular matrix with 1's on the diagonal can be factored into $E_i(x)$'s with $x \geq 0$. 
The following identities hold:

1. \( e_i (x) e_i (y) e_i (z) = e_i \left( \frac{x+y}{2} \right) e_i \left( \frac{z}{2} \right) \)
2. \( f_i (x) f_i (y) f_i (z) = f_i \left( \frac{x+y}{2} \right) f_i \left( \frac{z}{2} \right) \)

**Def.** A \( k \)-nonnegative matrix is a matrix where all minors of size \( \leq k \) are nonnegative.

\( k \)-nonnegative matrices are a semigroup for the same reasons as before.

**Problem:** What are the generators/relations for the semigroup of non-singular \( k \)-nonnegative matrices?

**Exercise:** For \( n=2, k=1 \), show the generators are \( e_1 (2), f_1 (2), \) diagonal matrices, and \( (01) \).

Another variation: **Restrict to upper triangular matrices with 1's on the diagonal.**