1. What's a matroid?

It abstracts a graph \( G = (V,E) \) and a matrix \( M = M_0 \) (thought of as their column vectors).

\[
\begin{bmatrix}
2a & b & c & d \\
\alpha & -1 & 0 & 1 \\
\beta & 0 & -1 & 0 \\
\gamma & 0 & 0 & -1
\end{bmatrix}
\]

To get \( M_0 \), arbitrarily orient edges.

Here, \( M_0 = [1,1,0,1] \).

The column vectors live in a 3-dimensional space.

Matroid only specifies the combinatorial data of (in)dependence, in various equivalent forms:

- **the bases** \( \mathcal{B}(M) = \{ \text{abcd, acd, bcd} \} \)

- **the independent sets** \( \mathcal{X}(M) = \{ \text{subsets of bases} \} \)

- **the circuits** \( \mathcal{C}(M) = \{ \text{minimal dependent sets under inclusion} \} \)

If \( M \) has no zero vectors (loops in \( G \)) and no (anti-)parallel vectors (multiple edges), then an equivalent way to specify the matroid is via

\( L(M) = \) lattice of flats \( F \subseteq E \) for \( M \), ordered by inclusion

\[
\text{subsets } F \text{ of vectors closed under linear span}
\]

\( i.e., \) closed under adding in vectors that don't increase \( r(F) := \max \{ |I: I \subseteq F, I \in \mathcal{I}(M) | \} \).

2. Characteristic polynomial

Turns out \( p_M(t) = t^{r(F)} \cdot \chi_M(t) \) is the characteristic polynomial of \( M \).

Where \( \chi_M(t) := \sum_{\text{flats } F \subseteq L(M)} \mu(F, F) t^{r(F)} \).

\( \mu(F, G) := +1 \) is the Mobius inversion.

EXII In our running example,

\[
\begin{align*}
\chi_M(t) &= (t-1) \chi_M(t) \\
&= (t-1)(t^2-3t+2)
\end{align*}
\]

In fact, \( \chi_M(t) = (t-1) \chi_M(t) \) reduced characteristic polynomial of \( M \),

\( \mu(0, F) \) is circled.
ExII L(M) helps abstract $p_6(t) = \text{chromatic polynomial} \quad \#	ext{ proper vertex } t \text{-colorings of } G_3^2$

Choose color for 2 first, t choices
then 3, t-1 choices
then 4, t-2 choices
then 0, t-1 choices

So $p_6(t) = t(t-1)^2(t-2)$
$= t^3 - 4t^2 + 5t - 2t$

coefficient always alternate in sign.

$p_F(t)$ has unsigned coefficient sequence 1, 4, 5, 2
- conjectured to be unimodal by Read 1968
conjectured log-concave, which
$(a_k \geq a_{k-1} a_{k+1})$ implies
by Hoggatt 1974.

2. Characteristic polynomial

Turns out

$\chi_M(t) = \prod_{\text{connected components } \chi_{M_i}(t)}$

where

$\chi_{M_i}(t) = \sum_{\text{flats } F \in L(M)} \mu(F, F') t^{r(F)} - r(F)$

$\mu(F, F') = -1$

EXII In our running example,

In fact, $\chi_M(t) = (t-1)\chi_M(t)$

reduced characteristic polynomial of $M$

$= (t-1)(t^2 - 3t + 2)$
the unsigned coefficient sequence was conjectured unimodal, log-concave for all matroids, by Rota - Heron - Welsh - Brylawski.

There was almost no progress until 2012...

3. The Chow ring $A(M)$ (Feichtner - Yuzvinsky, 2004)

$$A(M) = \mathbb{Z}[X_{\text{F}}]_{\text{non-empty}} / \left( X_{\text{F}} X_{\text{G}} : \text{F} \cap \text{G} \neq \emptyset \right) \cup \left( \alpha_i - \alpha_j : i \neq j \in E \right) \cup \left( \alpha_i - \beta_j : i \neq j \in E \right)$$

where $\alpha_i = \sum X_{\text{F}}$ if flat $\text{F}$ containing $i$

Exercise 20: In the above example, show $A(M)$ is a graded ring, $A(M) = A^0(M) \oplus A^1(M) \oplus A^2(M) \oplus \cdots$ with $A^i(M) : A^i(M) \subset A^{i+1}(M)$

(b) These elements in $A(M)$

$$x_i = x_i \quad y_i \in \mathbb{E}$$

$\beta_j = \sum x_F - \alpha_i$

are linear and satisfy $a^2 \to 1$ (is $\mu_0$)

$$a^2 \to -3, \quad x \to 6(\mu_1)$$

$$\beta^2 \to 2, \quad (z \mu_2)$$

Theorem (simple): (A.-H.-K., 2015) For matroid $M$,

(i) One always has $A(M) = A^0(M) \oplus A^1(M) \oplus \cdots \oplus A^{r-1}(M) \oplus A^r(M)$$

and $A^r(M) = 0$ for $k \geq r$.

(ii) $\alpha_i = \alpha_i \neq i \in E \quad \beta_j = \sum x_F$ \quad $f \neq i$

satisfy $a^{r-k} \beta^k \to \mu_k$ in $(\mu_0, \mu_1, \cdots, \mu_{r-1})$ = unsigned coefficients of $\overline{\mathbb{F}}$

(iii) One has isomorphisms $A^k(M) \to \mathfrak{A}^{r-k}(\mathbb{F})$ and working with $\mathbb{R}$ coefficients, $3$ elements $\beta_k \in A^k(M)$

It can be realized by $x \to \beta_k^{r-k} : x$

(iv) the quadratic forms $A^k(M) \to \mathfrak{A}^{r-k}(\mathbb{F})$ have predictable signatures.

(v) $\beta$ is a limit of ample $\beta_k$'s and somehow this implies $\mu_k \geq 2 \mu_{k-1} \mu_{k+1}$ (log-concavity?)

4. REU Problem 9 For various "favorite" matroids, e.g.

- $M = G = \mathbb{Z}_3$ complete graph $K_n$
- uniform (general) matroids $M_{n,r}$ of rank $r$, $1 \leq n$
- $M = \mathbb{Z}_3$ all vectors in $F_3^r$

(a) compute explicit formulas for the Hilbert series of $A(M)$:

$$1 + \dim A^0(M) \cdot x + \dim A^1(M) \cdot x^2 \cdots + \dim A^{r-2}(M) \cdot x^{r-2} + x^{r-1}$$

$$h = (1, \dim A^0(M), \ldots, \dim A^{r-2}(M), 1)$$

(b) compute $\beta$-vectors and Charney-Davis quantity for $h$.

(c) compute the eigenvalues of $Q_k = x : \beta^{r-k} \cdot x = x^T A x$, $A^r = A$

(d) When is $A(M)$ a Koszul algebra?

(e) What about the Smith normal forms of these $A_k$?
The Chow ring \( A(M) \) (Feichtner - Yuzvinsky, 2004)
\[
A(M) = \mathbb{Z}[X_{k}] / \left( \begin{array}{c}
X_{k} \cdot F \\
\text{non-empty, proper flats} \\
F \in L(M)
\end{array} \right)
\]
where \( k = \sum X_{k} \) flats \( F \) containing 1

\( M = \) \[
\begin{array}{cc}
\text{a} & \text{c} \\
\text{b} & \text{d}
\end{array}
\]

\( A(M) = \mathbb{Z}[X_{a}, X_{b}, X_{c}, X_{d}, X_{ad}, X_{bd}, X_{abc}] / \left( \begin{array}{c}
X_{a}X_{b} - X_{b}X_{a} \\
X_{a}X_{c} - X_{c}X_{a} \\
X_{a}X_{d} - X_{d}X_{a} \\
X_{c}X_{d} - X_{d}X_{c} \\
X_{abc} - X_{c}X_{d} - X_{d}X_{a} - X_{a}X_{c}
\end{array} \right)
\]

\( \lambda = 1 \)

\( A(M) = \mathbb{A}^{9}(M) \oplus \mathbb{A}^{1}(M) \oplus A^{2}(M) \oplus \ldots \)
with \( A^{2}(M) \cdot A^{1}(M) = A^{3}(M) \)

(b) These elements in \( A(M) \)
\[
\lambda = \sum \lambda_{i} \neq 1 \oplus \mathbb{L}
\]

\( \beta = \sum \lambda_{j} \neq \mathbb{L} \)

are linear and satisfy
\[
\begin{array}{c}
\lambda_{1} = 1 \\
\lambda_{2} = 0 \\
\lambda_{3} = 0
\end{array}
\]

Theorem(s) (A-H-K, 2015) (simple)
(i) One always has \( A(M) = A_{0}(M) \oplus A_{1}(M) \oplus A_{2}(M) \oplus \ldots \)
\[
\frac{\mathbb{A}^{2}(M)}{\mathbb{A}^{2}(M) \oplus A^{3}(M) \oplus A^{4}(M) \oplus \ldots}
\]

(zero for \( k \geq r \))

(ii) \( \lambda = \sum \lambda_{j} \neq 1 \oplus \mathbb{L} \)
\[
\beta = \sum \lambda_{j} \neq \mathbb{L}
\]

satisfy \( \alpha^{k-1} \beta = \alpha^{k} \lambda_{k} \)

(iii) One has isomorphisms \( A^{k}(M) \xrightarrow{\alpha^{k-1} \beta} \mathbb{A}^{2}(M) \)
\[A^{k}(M) \xrightarrow{\lambda_{k}} \mathbb{A}^{2}(M) \]

and working with \( \mathbb{Z} \) coefficients, \( A \) elements \( \beta \in A(M) \) s.t.
\[A^{k}(M) \xrightarrow{\lambda_{k}} \mathbb{A}^{2}(M) \]

it can be realized by \( x \mapsto \beta \cdot x \)

(iv) the quadratic forms \( A^{k}(M) \xrightarrow{Q \cdot x} \mathbb{A}^{2}(M) \)
\[x \mapsto Q \cdot x = \lambda_{k} \cdot x \]

have predictable signatures.

(v) \( \beta \) is a limit of ample \( \beta \)'s and somehow this implies \( \lambda_{k+1} = \lambda_{k} \cdot x \) (log-concavity!)

4. REU Problem 9
For various "favorite" matroids, e.g.
- \( M = M_{n} \) where \( G \) is complete graph \( K_{n} \)
- uniform (generic) matroids \( M_{n} \) of rank \( r \), \( 1 \leq r \)
- \( M = \mathbb{F} \) all vectors in \( \mathbb{F}^{2} \)

(a) compute explicit formulas for the Hilbert series of \( A(M) \):
\[1 + \dim A^{1}(M) + \dim A^{2}(M) + \ldots + \dim A^{r-1}(M) + \dim A^{r}(M) \]
\[h = 1, \dim A^{1}(M), \ldots, \dim A^{r-1}(M), 1 \]
\[h_{1}, h_{r-1}, h_{r-2}, \ldots, h_{r}
\]

(b) compute \( \lambda \)-vectors and Charny-Davis quantity for \( h \).
(c) compute the eigenvalues of \( Q \cdot x = x^{T} \lambda \cdot x \).
(d) When is \( A(M) \) a Koszul algebra?
(e) What about the Smith normal forms of these \( A \) ?