1. What is a matroid?

A matroid abstracts a graph \( G = (V, E) \) and a matrix \( M = M_G \), thought of as column vectors,

\[
\begin{bmatrix}
a & b & c & d \\
1 & +1 & -1 & 0 & +1 \\
2 & -1 & 0 & +1 & 0 \\
3 & 0 & +1 & -1 & 0 \\
4 & 0 & 0 & 0 & -1 \\
\end{bmatrix}
\]

by only specifying these equivalent bits of combinatorial data:

- **the bases** \( \mathcal{B}(M) = \{ \text{bd, ac, bc, cd} \} \ (\leftrightarrow \text{spanning forests of } G) \)

- **the independent sets** \( \mathcal{I}(M) = \{ \text{subsets of bases} \} \) \( \leftrightarrow \text{independent sets in } G \)

- **the circuits** \( \mathcal{C}(M) = \{ \text{minimal dependent sets} \} = \{ \text{abc} \} \ (\leftrightarrow \text{cycles in } G) \)

If \( M \) has no zero vectors (loops in \( G \)) and no parallel vectors (parallel edges in \( G \)), then one can also specify...

\[\bigcirc\]
\( \mathcal{L}(M) \) : lattice of flats \( F \leq E \), ordered by \( \subseteq \)
(subsets of vectors closed under linear span
i.e. closed under adding vectors that don't increase rank \( r(A) \)
where \( r(A) := \max \{ t | I \subseteq ICA_0 \} \)

\[
\begin{align*}
\mu(\phi, E) \text{ values circled} \\
\end{align*}
\]

Example: This helps abstract the chromatic polynomial \( P_G(t) \) = \#proper vertex-colorings with \( t \) colors

\[
\begin{align*}
P_G(t) &= t(t-1)(t-2)(t-3) \\
&= t^4 - 4t^3 + 5t^2 - 2t \\
\end{align*}
\]

Unsigned coefficients \( (1, 4, 5, 2) \)
log-concave: \( 4, 21, 52, 80 \)

2. Characteristic polynomial

\[
\begin{align*}
\text{Turns out } P_G(t) &= t \chi_M(t) \\
\chi_M(t) &= \sum_{ \text{flats } F \in \mathcal{L}(M) } \mu(\phi, F) t^r \\
\mu(\phi, F) &= +1 \\
\mu(\phi, F) &= - \sum \mu(\phi, G) \\
\end{align*}
\]

Möbius function

\[
\begin{align*}
\mu(\phi, \phi) &= +1 \\
\mu(\phi, F) &= - \sum \mu(\phi, G) \\
G \leq F \\
\end{align*}
\]

Reduced characteristic polynomial

\[
\begin{align*}
\chi_M(t) &= t^4 - 4t^3 + 5t^2 - 2t \\
&= (t-1)(t^3 - 3t + 2) \\
&= (t-1)(t^2 - 3t + 2) \\
\end{align*}
\]

Conjectured unimodal, log-concave by Rota-Heron-Welsh-Bielawski 1997

1931, 1932, 1937

log-concave: \( 3, 9, 21, 39, 69 \)

Almost no progress for 35 years! Until...
3. The Chow ring of $M$ (Friedman-Yuzvinsky 2004)

$$A(M) := \mathbb{Z}[X_F]_{F \text{ a non-empty proper flat of } M} \left/ \left( \bigoplus_{G \subseteq F} (X_F X_G : F \not\subseteq F) + (\alpha_i - \alpha_j : i \neq j \in F) \right) \right.$$ 

where $\alpha_i := \sum_{F \ni i} X_F$.

**Example:** $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has degree 1.

$$A(M) = \mathbb{Z}[X_a, X_b, X_c, X_d, X_{ab}, X_{ad}, X_{bd}, X_{cd}] \left/ \left( X_a X_b, X_a X_c, X_a X_d, X_b X_c, X_b X_d, X_c X_d, X_{ab} X_{bc}, X_{ad} X_{bd}, X_{cd} X_{bc}, X_a + X_b + X_c + X_d \right. \right.$$ 

Since the relations $X_a X_d = 0$, $\alpha_i = \alpha_j$ are homogeneous,

$$A(M) = \bigoplus_{n=1}^{\infty} A^n(M) \otimes \mathbb{Z}$$

is a graded ring, i.e., $A^i(M) \subseteq A^{i+1}(M) \otimes \mathbb{Z}$

**REU Exercise 20:** In the above example, show

(a) $A(M) = A^0(M) \oplus A^1(M) \oplus A^2(M) \oplus \cdots$, i.e., $A^3(M) = A^4(M) = \cdots = 0$

(b) with an isomorphism $A^2(M) \cong \mathbb{Z}$ sending $X_{F_1} X_{F_2} \mapsto 1$

<table>
<thead>
<tr>
<th>$F_1 \not\subseteq F_2$</th>
<th>$X_a X_{ad}$</th>
<th>$X_a X_{bc}$</th>
<th>$X_a X_{bd}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X_a^2$</td>
<td>$-1$</td>
<td>$X_{ad}$</td>
<td>$X_{bc}$</td>
</tr>
<tr>
<td>$X_{ad}$</td>
<td>$-1$</td>
<td>$X_{bc}$</td>
<td>$X_{bd}$</td>
</tr>
<tr>
<td>$X_{bc}$</td>
<td>$-1$</td>
<td>$X_{bd}$</td>
<td>$X_{ad}$</td>
</tr>
</tbody>
</table>

(b) The elements $\alpha_1 = X_a X_{ad}$ and $\beta = \sum_{i} X_i - \alpha$ satisfy $\deg(\alpha^2) = 1$ ($\leq 1$), $\deg(\alpha \beta) = 3$ ($\leq 1$), $\deg(\beta^2) = 2$ ($\leq 1$).
THEOREM(S) (Adiprasito-Huh-Katz, 2015) For a matroid $M$ of rank $r$,

\[ A(M) = \bigoplus_{i=0}^{r} A^i(M) \]

(i) One always has $A(M) = \mathbb{Z}/\mathbb{Z}_{\text{maximal flags}}$.

(ii) The elements $x_i = x_i: \forall i \in F$ and $\beta^i = \sum_{\text{all non-projective flags}} x_F - x_i$ satisfy $\mu_k = \deg(x_i^{r-1-k} \beta^i)$ where $\mu_0, ..., \mu_2 \in \text{coefficients of } A(M)$. (iii) One has isomorphisms $A^k(M) \longrightarrow A^{r-1-k}(M)$ for all $k \leq r-1$.

and working with $\mathbb{R}$ coefficients, they can be realized explicitly via certain ample elements $\beta^i \in A^i(M)$ as $x \longmapsto \beta^i \cdot x$.

(iv) The quadratic forms $A^k(M) \longrightarrow \mathbb{R}$ have easily predictable signature for ample $\beta^i$.

(v) $\beta$ is a limit of ample $\beta^i$'s, leading to $M_{k+1}^2 \geq M_k$. And the log-concavity conjectures are not obvious why this should be!
REU PROBLEM 9:

For various "favorite" matroids, e.g.
- $M = M_G$ where $G$ = complete graph $K_n$
- uniform (= generic) matroids $\bigcup_{n=5}^m \binom{K_n}{\text{rank}}$
- $M = \{\text{all vectors in } F_q^n\}$

(a) compute explicit formulas for the Hilbert series

$$1 + \frac{\dim A_1(M)}{b_1} t + \frac{\dim A_2(M)}{b_2} t^2 + \ldots + \frac{\dim A_{r-1}(M)}{b_{r-1}} t^{r-2} + t^r$$

(b) compute $x$-vectors and Channey-Davis quantities

for $(b_0, b_1, \ldots, b_{r-1})$

(keywords!)

(c) compute the eigenvalues of the quadratic forms $Q(x) = x \cdot B \cdot x$

on $A_r(M)$

(d) what about the Smith normal forms

for their associated symmetric matrices $A = A^T$

(e) when is $A(M)$ a Koszul algebra?

Try this in Sage Cell Server:

```python
G = Graph([[(1,2,'a'), (3,1,'b'), (2,3,'c'), (1,4,'d')]])
M = Matroid(G);
print(list(M.bases()));
for r in [0,1,2,3]:
    print(list(M.flats(r)));
AM = M.chow_ring();
AM.gens();
```