Equality of Schur Supports of Ribbons

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Overview

1. Preliminaries

2. Main Results

3. Other Results

4. Acknowledgements
Schur Functions

Example/Definition (Young Diagram & Semistandard Young Tableau)

Partition $\lambda = (4, 3, 2)$

Young Diagram

SSYT

Definition

The Schur function $s_\lambda$ of a partition $\lambda$ is

$$s_\lambda(x_1, x_2, x_3, \ldots) = \sum_{T : \text{SSYT of shape } \lambda} x^T = \sum_{T} x_1^{t_1} x_2^{t_2} x_3^{t_3} \ldots$$

where $t_i$ is the number of occurrences of $i$ in $T$. 
Skew Schur Functions

Example/Definition (Skew Shape)

\[ \lambda = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \quad \mu = \begin{array}{cc} \square \end{array} \quad \lambda/\mu = \begin{array}{cccc} \square & \square & \square & \square \\ \square & \square & \square & \square \\ \square & \square & \square & \square \end{array} \]

Skew Schur functions are defined analogously to straight Schur functions. Skew-positive, meaning

\[ s_{\lambda/\mu} = \sum_{\nu} c_{\lambda,\mu,\nu}s_{\nu} \]

where \( \nu \) is a straight partition, and \( c_{\lambda,\mu,\nu} \geq 0 \).

Definition

The Schur support of a skew shape \( \lambda/\mu \), denoted \( [\lambda/\mu] \), is defined as

\[ [\lambda/\mu] = \{ \nu : c_{\lambda,\mu,\nu} > 0 \} \]
Skew Schur Functions

Example/Definition (Skew Shape)

\[ \lambda = \begin{array}{cccc}
\text{■} & \\
\text{■} & \\
\text{■} & \\
\text{■} & \\
\text{■} & \\
\end{array} \quad \mu = \begin{array}{cc}
\text{□} & \\
\end{array} \quad \lambda/\mu = \begin{array}{cccc}
\text{□} & \\
\text{□} & \\
\text{□} & \\
\text{□} & \\
\end{array} \]

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Skew Schur Functions

Example/Definition (Skew Shape)

\[
\lambda = \begin{array}{cccc}
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\square & \square & \square & \square \\
\end{array} \quad \mu = \begin{array}{llll}
\square & \square & \square & \\
\end{array} \quad \lambda/\mu = \begin{array}{cccc}
\square & \square & \square & \\
\square & \square & \square & \\
\square & \square & \square & \\
\square & \square & \square & \\
\end{array}
\]

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\]
Ribbons

A ribbon is a skew shape which does not contain a $2 \times 2$ subdiagram.

Example

Given a sequence of integers, there’s a unique ribbon with that sequence of row lengths. Thus, ribbons are uniquely determined by compositions of $n$ (the total boxes).

The above example can be denoted as the ribbon $(3, 2, 3)$
Littlewood-Richardson Rule

This is a rule to check if a particular straight young diagram is present in the support.

Formal Definition

Let \( D \) be a skew shape. A partition \( \lambda = (\lambda_1, \ldots, \lambda_m) \) is in the support of \( s_D \) iff there is a valid LR-filling of \( D \) with content \( \lambda \).

A filling of \( D \) is an LR-filling if:

- The tableau is semistandard.
- Every initial reverse reading word is Yamanouchi:
  \[ #i's \geq #(i + 1)'s \]
Littlewood-Richardson Rule

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**Formal Definition**

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  $\#i$’s $\geq \#(i + 1)$’s

**Example/Definition (Yamanouchi Property)**

Reverse Reading Word: 1,1,1,2,2,3,2

This is Yamanouchi because there are at least as many 1’s as 2’s and as many 2’s as 3’s at every stage.
Littlewood Richardson Rule

Example of LR-rule

Reverse Reading Word: 1,1,2,2,1,3,2

This is Yamanouchi and semistandard, hence is a valid LR-filling

The content of the filling is (3,3,1), thus (3,3,1) is in the support of the ribbon: (2,3,2).
Littlewood Richardson Rule

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Reverse Reading Word: 1,1,2,2,1,3,2

This is Yamanouchi and semistandard, hence is a valid LR-filling

The content of the filling is (3,3,1), thus (3,3,1) is in the support of the ribbon: (2,3,2).

Proposition

Let $\alpha = (1, \alpha_2, \alpha_3)$ be a ribbon. Then, $\alpha' = (\alpha_2, 1, \alpha_3)$ and $\alpha$ don’t have the same support.

Proof. When the row of length 1 is in the middle, there is no LR-filling with just 1’s and 2’s.

\[\alpha = (1, 2, 2) \quad \alpha' = (2, 1, 2)\]
[McNamara (2008)] gives that 2 ribbons can have the same support only if one is a permutation of the rows of the other.

Definition

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a ribbon. We use $\alpha_\pi$ to denote a ribbon formed by applying the permutation $\pi \in S_m$ to the row lengths of $\alpha$. 

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**Equality of Schur Supports of Ribbons**

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Preliminaries

Main Results

Other Results

Acknowledgements

References
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![Diagram of ribbons](attachment:ribbons_diagram.png)
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**Definition**

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \) be a ribbon. We use \( \alpha_\pi \) to denote a ribbon formed by applying the permutation \( \pi \in S_m \) to the row lengths of \( \alpha \).

\[
\begin{align*}
\alpha &= \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array} \\
\alpha_{(2 \ 3)} &= \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array} \\
\alpha_{(1 \ 2)} &= \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array} \\
\alpha_{(1 \ 3 \ 2)} &= \begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
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1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
\end{array}
\end{align*}
\]

**Definition**

A ribbon \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \) is said to have **full equivalence class** if for any permutation \( \pi \in S_m \), we have \([\alpha] = [\alpha_\pi] \).
Trivial Cases

Proposition

If a ribbon $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ has $k$ rows of length 1, where $1 \leq k < m$, then $\alpha$ does not have full equivalence class.
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If a ribbon $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ has $k$ rows of length 1, where $1 \leq k < m$, then $\alpha$ does not have full equivalence class.

Remark

It is well known that rotating a ribbon by $180^\circ$ preserves its support. It follows trivially that any ribbon with only two rows has full equivalence class.

For the rest of the presentation, we consider only ribbons with more than two rows and with no rows of length 1.
Sufficient Condition for Full E.C

Theorem (S, G, H, Tran, '17)

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \) be a ribbon such that any subset of size three of \( \{\alpha_i\} \) satisfies the **strict** triangle inequality \( \alpha_i < \alpha_j + \alpha_k \). Then \( \alpha \) has full equivalence class.

**Proof Idea:** Given a ribbon with an LR-filling, show how to swap two adjacent row lengths while preserving the content, Yamanouchi property, and semistandardness of the filling.
Theorem (S, G, H, Tran, ’17)

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \) be a ribbon such that any subset of size three of \( \{\alpha_i\} \) satisfies the \textbf{strict} triangle inequality (\( \alpha_i < \alpha_j + \alpha_k \)). Then \( \alpha \) has full equivalence class.

**Proof Idea:** Given a ribbon with an LR-filling, show how to swap two adjacent row lengths while preserving the content, Yamanouchi property, and semistandardness of the filling.

**Proof Sketch:**

1. Use the \( R \)-matrix algorithm (described on the next slide) to swap adjacent row lengths while preserving content and the Yamanouchi property.

2. Show how to adjust the resulting filling to be semistandard.
Algorithm [Inoue et al. (2012), Section 2.2.3]

1. Represent the rows to be swapped as box-ball systems.

\[
R \begin{pmatrix} 1 & 3 & 3 & 4 & 7 \otimes 1 & 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 7 \otimes 1 & 3 & 3 & 3 & 5 \end{pmatrix}
\]
**Algorithm [Inoue et al. (2012), Section 2.2.3]**

1. Represent the rows to be swapped as box-ball systems.
2. For each unconnected ball $A$ on the right, find its partner $B$ on the left which is an unconnected ball in the lowest position but higher than that of $A$; if there are no such balls, choose from the balls in the lowest position on the left. Connect $A$ and $B$.

\[
R\left( \begin{array}{cccc} 1 & 3 & 3 & 4 & 7 \\ \otimes & & & & \\ \end{array} \right) = \begin{array}{cccc} 1 & 4 & 7 \\ \otimes & & & \end{array} \begin{array}{cccc} 1 & 3 & 3 & 3 & 5 \\ \end{array}
\]
Algorithm [Inoue et al. (2012), Section 2.2.3]

1. Represent the rows to be swapped as box-ball systems.
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3. Shift all unconnected balls from the left to the right.

\[
R\left( \begin{array}{cc}
1 & 3 \\
3 & 4 \\
7 & \\
\end{array} \right) \otimes \begin{array}{c}
1 \\
3 \\
5 \\
\end{array} = \begin{array}{cc}
1 & 4 \\
7 & \\
\end{array} \otimes \begin{array}{ccccc}
1 & 3 & 3 & 3 & 5 \\
\end{array}
\]
In this example, notice that

1. the Yamanouchi property is preserved.
2. the leftmost entry in the bottom row does not increase.

\[
\begin{array}{cccccc}
1 & 3 & 3 & 4 & 7 \\
1 & 3 & 5
\end{array} \longrightarrow \begin{array}{cccc}
1 & 3 & 3 & 3 \\
1 & 4 & 7 & 5
\end{array}
\]

In fact, we prove that (1) and (2) hold in general. The remainder of the proof of the theorem ensures that we can move around the content within the ribbon so that the rightmost entry in the top row does not violate semistandardness.
Necessary Condition for Full E.C

**Theorem (S, G, H, Tran, ’17)**

Let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \) be a ribbon, where \( \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m \). If \( \alpha \) has full equivalence class, then \( N_j < \sum_{i=j+1}^{m} \alpha_i - (m - j - 2) \) for all \( j \leq m - 2 \), where

\[
N_j = \max \{ k | \sum_{i \leq j: \alpha_i \leq k} (k - \alpha_i) \leq m - j - 2 \}.
\]
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\]

We prove the contrapositive by assuming

\[
N_j \geq \sum_{i=j+1}^{m} \alpha_i - (m - j - 2)
\]

and showing that there exists a content for an LR-filling of \( \alpha(j, j+1) \) that is not the content of any LR-filling of \( \alpha \).
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N_j = \max\{ k \mid \sum_{i \leq j: \alpha_i < k} (k - \alpha_i) \leq m - j - 2 \}.
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We prove the contrapositive by assuming

\[
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Conjecture

The above necessary condition is sufficient for a ribbon to have full equivalence class.
Other Results

Proposition: 3 rows

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ be a ribbon. Then $\alpha$ has full E.C iff $\alpha_1, \alpha_2$ and $\alpha_3$ satisfy the strict triangle inequality.

Proposition: 4 rows

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a ribbon such that $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$. Then, $\alpha$ has full equivalence class iff

- $\alpha_1 < \alpha_2 + \alpha_3 + \alpha_4 - 2$
- $\alpha_2 < \alpha_3 + \alpha_4$
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References


