A Rule of Three for Schur Q-functions

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Outline

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Notation

\(e_k\)'s are elementary symmetric polynomials:

\[
e_k(x_1, \cdots, x_n) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} x_{j_k} \cdots x_{j_1}
\]

\(h_k\)'s are homogeneous symmetric polynomials:

\[
h_k(x_1, \cdots, x_n) = \sum_{1 \leq j_1 \leq \cdots \leq j_k \leq n} x_{j_1} \cdots x_{j_k}
\]

For \(x_1, \ldots, x_n\) and \(S \subset [n]\) with \(S = \{s_1 < s_2 < \cdots < s_k\}\):

\[
[x, y] = xy - yx
\]

\[
e_i(x_S) = e_i(x_{s_1}, \ldots, x_{s_k})
\]

\[
x_S = x^\downarrow_S = x_{s_k} x_{s_{k-1}} \cdots x_{s_1}
\]
What is a Rule of Three?

Kirillov 2016:

**Theorem (1.1)**

For \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \) tuples of elements in a ring \( R \), the following are equivalent:

- \( [e_k(u_S), e_\ell(v_S)] = 0 \) for any \( k, \ell, S \subset [n] \),
- the above holds for \( |S| \leq 3 \) and \( k\ell \leq 3 \); that is,

\[
\begin{align*}
[e_1(u_S), e_1(v_S)] &= 0 \\
[e_1(u_S), e_2(v_S)] &= 0 \\
[e_2(u_S), e_1(v_S)] &= 0 \\
[e_1(u_S), e_3(v_S)] &= 0 \\
[e_3(u_S), e_1(v_S)] &= 0
\end{align*}
\]
Consider $u = v = (a, b, c)$. Then we have the following relations for $S = \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

\[
\begin{align*}
[e_1(u_S), e_1(v_S)] &= 0 \\
[e_1(u_S), e_2(v_S)] &= 0 \\
[e_2(u_S), e_1(v_S)] &= 0 \\
[e_1(u_S), e_3(v_S)] &= 0 \\
[e_3(u_S), e_1(v_S)] &= 0
\end{align*}
\]
Example (cont.)

\[(a + b)(ba) - (ba)(a + b)\]
\[(a + c)(ca) - (ca)(a + c)\]
\[(b + c)(cb) - (cb)(b + c)\]
\[(a + b + c)(cb + ca + ba) - (cb + ca + ba)(a + b + c)\]
\[(a + b + c)(cba) - (cba)(a + b + c)\]
Example (cont.)

\[\begin{align*}
\text{aba} + \text{bba} - \text{baa} - \text{bab} \\
\text{aca} + \text{cca} - \text{caa} - \text{cac} \\
\text{bcb} + \text{ccb} - \text{cbb} - \text{cbc} \\
\text{acb} + \text{bca} - \text{cab} - \text{bac} \\
\text{acba} + \text{caba} - \text{cbca} - \text{cbac}
\end{align*}\]

These must generate all of \([e_k(u_S), e_\ell(u_S)]\).
Fomin-Greene 2006:

**Theorem**

If nonadjacent variables commute (or satisfy the non-local Knuth relations) and adjacent variables \( a < b \) satisfy

\[
[e_1(a, b), e_2(a, b)] = 0
\]

then noncommutative Schur functions behave as if they were ordinary Schur functions.
More Rules of Three

Blasiak and Fomin 2016 generalized to:

- Super elementary symmetric polynomials
- Generating functions over rings
- Sums and products
Problem

Can we use this theory to give rules of three in other settings?
Definition by Analogy

Schur functions : semistandard Young tableaux ::
Schur Q-functions : semistandard shifted Young tableaux
Definition
A (semistandard) shifted Young tableau $T$ of shape $\lambda$ is a filling of a shifted diagram $\lambda$ with letters from the alphabet $A = \{1' < 1 < 2' < 2 < \cdots\}$ such that:

- Rows and columns are weakly increasing;
- Each column has at most one $k$ for $k \in \{1, 2, \cdots\}$;
- Each row has at most one $k$ for $k \in \{1', 2', \cdots\}$.

Example
For $\lambda = (5, 4, 2)$, a possible tableau:

```
1 2' 3' 3 3
2' 3 3 4' 4
4' 4
```
Examples

\[
Q_1(x_1, x_2) = x_1 + x_1' + x_2 + x_2' = 2x_1 + 2x_2
\]

\[
Q_2(x_1, x_2) = x_{11} + x_{1'1} + x_{22} + x_{2'2}' + x_{12} + x_{1'2} + x_{12'} + x_{1'2'} = 2x_1^2 + 2x_2^2 + 4x_1x_2
\]

\[
= \frac{1}{2} Q_1^2
\]
Properties

- $Q_{\lambda}$ are symmetric;
- $\mathbb{Q}[Q_{\lambda}]$ form the same subalgebra as $\mathbb{Q}[p_{2k+1}]$, where $p_a = \sum x_i^a$;
- $\mathbb{Q}[Q_{\lambda}] = \mathbb{Q}[Q_{(2k+1)}]$. 
Non-commutative Case

How do we generalize?

\[
\begin{array}{c|c|c|c|c|c}
1' & 1 & 2' & 3 & 3' & 4 & 5' \\
\end{array}
\xrightarrow{\text{ }}
\begin{array}{c}
x_5 x_4 x_3 x_3 x_2 x_1 x_1 \\
\end{array}
\text{(descending)}

\xleftarrow{\text{ }}
\begin{array}{c}
x_5 x_4 x_2 x_1 x_1 x_3 x_3 x_4 \\
\end{array}
\text{(hook)}
Non-commutative Case

How do we generalize?

\[
\begin{array}{c|c|c|c|c|c|c}
1' & 1 & 2' & 3 & 3' & 4 & 5' \\
\end{array}
\quad \leftrightarrow \quad x_5 x_4 x_4 x_3 x_3 x_2 x_1 x_1 (\text{descending})
\quad \leftrightarrow \quad x_5 x_4 x_2 x_1 x_1 x_3 x_3 x_4 (\text{hook})
\]

\[
Q^{(2)}(x_1, x_2) = x_1 x_1 + x_1' x_1 + x_2 x_2 + x_2' x_2
\]
\[
+ x_1 x_2 + x_1' x_2 + x_2 x_1 + x_2' x_1
\]
\[
= 2x_1^2 + 2x_2^2 + 4x_2 x_1 (\text{descending})
\]
\[
= 2x_1^2 + 2x_2^2 + 2x_2 x_1 + 2x_1 x_2 (\text{hook})
\]
\[
= \frac{1}{2} Q^{(1)}
\]

Hook reading is the more natural.
Proposed Rule of Three

Conjecture

Let $u_1, \ldots, u_N, v_1, \ldots, v_N$ be elements of a ring $A$. The following are equivalent:

- $Q(k)(u_S)$ and $Q(\ell)(v_S)$ commute for all $S, k, \ell$.
- the above holds when $k = 1$ or $\ell = 1$ (for all $S$)

Computation suggests this is optimal.
Naive Approach

\[
\begin{align*}
aba + bba - baa - bab \\
aca + cca - caa - cac \\
bcb + ccb - cbb - cbc \\
acb + bca - cab - bac \\
acba + caba - cbca - cbac
\end{align*}
\]

Can we get the next simplest commutation relation?

\[
\begin{align*}
C' &= \left[ e_2(a, b, c), e_3(a, b, c) \right] \\
    &= (cb + ca + ba)(cba) - (cba)(cb + ca + ba)
\end{align*}
\]
Yes!

\[ C' = (bcb + ccb - cbb - cbc)(aa + ab - ba) \\
    + (cc + ac + bc - cb - ca)(aba + bba - ba a - bab) \\
    - (aca + cca - caa - cac)ba \\
    - (acb + bca - cab - bac)ba \\
    + (acba + caba - cbca - cbac)(a + b) \]
In the commutative case:

\[ a_i = 1 + xu_i \]
\[ b_i = (1 - xu_i)^{-1} \]
\[ q_i = a_ib_i = (1 + xu_i)(1 - xu_i)^{-1} \]

And:

\[ a[n] = \sum e_kx^k \]
\[ b[n] = \sum h_kx^k \]
\[ q[n] = \sum Q(k)x^k \]
In the non-commutative case:

\[ a_i = 1 + xu_i \]
\[ b_i = (1 - xu_i)^{-1} \]
\[ q_i = a_i b_i = (1 + xu_i)(1 - xu_i)^{-1} \]

And:

\[ a_{[n]}^\uparrow = \sum e_k x^k \]
\[ b_{[n]}^\uparrow = \sum h_k x^k \]
\[ q_{[n]}^\downarrow = \sum Q_{(k)} x^k \text{ descending reading} \]
In the *non-commutative* case:

\[ a_i = 1 + xu_i \]
\[ b_i = (1 - xu_i)^{-1} \]

And:

\[ a_{\downarrow}[n] = \sum e_k x^k \]
\[ b_{\uparrow}[n] = \sum h_k x^k \]
\[ a_{\downarrow}[n] b_{\uparrow}[n] = \sum Q(k) x^k \text{  hook reading} \]
Theorem (Blasiak-Fomin 3.5)

Let $R$ be a ring, and let $g_1, \ldots, g_N, h_1, \ldots, h_N \in R$ be potentially invertible elements. Then the following are equivalent:

- $\left[ \sum_{i \in S} g_i, \sum_{i \in S} h_i \right] = 0,$
- $\left[ \sum_{i \in S} g_i, h_S \right] = 0,$
- $\left[ g_S, \sum_{i \in S} h_i \right] = 0,$
- $\left[ g_S, h_S \right] = 0$ for all subsets $S$.
- the above holds for $|S| \leq 3$.

Use $g_i = 1 + x u_i, h_i = 1 + y v_i$ to get rule of three for $e_k$'s.
Conjecture

Let $A$ be a ring, and let $\{x_i\}_{i \in [N]}, \{y_i\}_{i \in [N]} \in A$. Then define

$$a_i = 1 + x_i t \quad \quad \quad \beta_i = 1 - y_i s$$

$$b_i = 1 - x_i t \quad \quad \quad \quad \alpha_i = 1 + y_i s$$

Further, let the following be true.

$$\left[ \sum_{i \in S} x_i, \alpha_S (\beta_S)^{-1} \right] = 0$$

$$\left[ \sum_{i \in S} y_i, a_S (b_S)^{-1} \right] = 0$$

Then $a_{[N]} (b_{[N]})^{-1} \alpha_{[N]} (\beta_{[N]})^{-1} = \alpha_{[N]} (\beta_{[N]})^{-1} a_{[N]} (b_{[N]})^{-1}$. 
Proof Progress

We have been attempting to replicate Blasiak and Fomin’s proof of Lemma 8.2, both in the **standard case**, and in the **weakened case** where nonadjacent variables commute.

- □: proof for the conjecture;
- △: proof for the conjecture for $|S| = 2$.

Arrows represent dependencies.
Further Questions

- Can we prove the conjecture? What about in a weaker setting?
- When does commutativity extend to all Schur Q-functions?
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