A Rule of Three for Schur Q-functions

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Outline

Notation

e_ks are elementary symmetric polynomials:

$$e_k(x_1, \cdots, x_n) = \sum_{1 \le j_1 < \cdots < j_k \le n} x_{j_k} \cdots x_{j_1}$$

h_ks are homogeneous symmetric polynomials:

$$h_k(x_1, \cdots, x_n) = \sum_{1 \le j_1 \le \cdots \le j_k \le n} x_{j_1} \cdots x_{j_k}$$

For $x_1, \ldots, x_n$ and $S \subset [n]$ with $S = \{s_1 < s_2 < \cdots < s_k\}$:

$$\begin{aligned} [x, y] &= xy - yx \\ e_i(x_S) &= e_i(x_{s_1}, \ldots, x_{s_k}) \\ x_S = x_S^{\downarrow} &= x_{s_k} x_{s_{k-1}} \cdots x_{s_1} \end{aligned}$$

What is a Rule of Three?

Kirillov 2016:

Theorem (1.1)

For $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ tuples of elements in a ring R, the following are equivalent:

- $[e_k(u_S), e_\ell(v_S)] = 0$ *for any* $k, l, S \subset [n]$,
- *the above holds for* $|S| \leq 3$ *and* $kl \leq 3$; *that is,*

$$\begin{aligned}
[e_1(u_S), e_1(v_S)] &= 0 \\
[e_1(u_S), e_2(v_S)] &= 0 \\
[e_2(u_S), e_1(v_S)] &= 0 \\
[e_1(u_S), e_3(v_S)] &= 0 \\
[e_3(u_S), e_1(v_S)] &= 0
\end{aligned}$$

Example

Consider $u = v = (a, b, c)$. Then we have the following relations for $S = \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}$.

$$[e_1(u_S), e_1(v_S)] = 0$$
$$[e_1(u_S), e_2(v_S)] = 0$$
$$[e_2(u_S), e_1(v_S)] = 0$$
$$[e_1(u_S), e_3(v_S)] = 0$$
$$[e_3(u_S), e_1(v_S)] = 0$$

Example (cont.)

$$(a+b)(ba)-(ba)(a+b)$$
$$(a+c)(ca)-(ca)(a+c)$$
$$(b+c)(cb)-(cb)(b+c)$$
$$(a+b+c)(cb+ca+ba)-(cb+ca+ba)(a+b+c)$$
$$(a+b+c)(cba)-(cba)(a+b+c)$$

Example (cont.)

$$aba + bba - baa - bab$$
$$aca + cca - caa - cac$$
$$bcb + ccb - cbb - cbc$$
$$acb + bca - cab - bac$$
$$acba + caba - cbca - cbac$$

These must generate all of $[e_k(u_S), e_\ell(u_S)]$.

Motivation

Fomin-Greene 2006:

Theorem

If nonadjacent variables commute (or satisfy the non-local Knuth relations) and adjacent variables $a < b$ satisfy

$$[e_1(a,b), e_2(a,b)] = 0$$

then noncommutative Schur functions behave as if they were ordinary Schur functions.

More Rules of Three

Blasiak and Fomin 2016 generalized to:

- Super elementary symmetric polynomials
- Generating functions over rings
- Sums and products

Research Problem

Problem

Can we use this theory to give rules of three in other settings?

Definition by Analogy

Schur functions : semistandard Young tableaux ::

Schur Q-functions : semistandard shifted Young tableaux

Definition

A *(semistandard) shifted Young tableau* T of shape λ is a filling of a shifted diagram λ with letters from the alphabet $A = \{1' < 1 < 2' < 2 < \cdots\}$ such that:

- Rows and columns are weakly increasing;
- Each column has at most one k for $k \in \{1, 2, \cdots\}$;
- Each row has at most one k for $k \in \{1', 2', \cdots\}$.

Example

For $\lambda = (5, 4, 2)$, a possible tableau:

1	$2'$	$3'$	3	3
	$2'$	3	$4'$	4
		$4'$	4	

Examples

$$
\begin{aligned}
Q_{(1)}(x_1, x_2) &= x^{\boxed{1}} + x^{\boxed{1'}} + x^{\boxed{2}} + x^{\boxed{2'}} \\
&= 2x_1 + 2x_2 \\
Q_{(2)}(x_1, x_2) &= x^{\boxed{1}\boxed{1}} + x^{\boxed{1'}\boxed{1}} + x^{\boxed{2}\boxed{2}} + x^{\boxed{2'}\boxed{2}} \\
&\quad + x^{\boxed{1}\boxed{2}} + x^{\boxed{1'}\boxed{2}} + x^{\boxed{1}\boxed{2'}} + x^{\boxed{1'}\boxed{2'}} \\
&= 2x_1^2 + 2x_2^2 + 4x_1x_2 \\
&= \frac{1}{2} Q_{(1)}^2
\end{aligned}
$$

Properties

- Q_λ are symmetric;
- $\mathbb{Q}[Q_\lambda]$ form the same subalgebra as $\mathbb{Q}[p_{2k+1}]$, where $p_a = \sum x_i^a$;
- $\mathbb{Q}[Q_\lambda] = \mathbb{Q}[Q_{(2k+1)}]$.

Non-commutative Case

How do we generalize?

$$\boxed{1'}\boxed{1}\boxed{2'}\boxed{3}\boxed{3}\boxed{4'}\boxed{4}\boxed{5'} \longleftrightarrow x_5x_4x_4x_3x_3x_2x_1x_1 \text{ (descending)}$$

$$\longleftrightarrow x_5x_4x_2x_1x_1x_3x_3x_4 \text{ (hook)}$$

Non-commutative Case

How do we generalize?

$$\boxed{1'}\boxed{1}\boxed{2'}\boxed{3}\boxed{3}\boxed{4'}\boxed{4}\boxed{5'} \longleftrightarrow x_5x_4x_4x_3x_3x_2x_1x_1 \text{ (descending)}$$

$$\longleftrightarrow x_5x_4x_2x_1x_1x_3x_3x_4 \text{ (hook)}$$

$$\begin{aligned} Q_{(2)}(x_1, x_2) &= x^{\boxed{1}\boxed{1}} + x^{\boxed{1'}\boxed{1}} + x^{\boxed{2}\boxed{2}} + x^{\boxed{2'}\boxed{2}} \\ &\quad + x^{\boxed{1}\boxed{2}} + x^{\boxed{1'}\boxed{2}} + x^{\boxed{1}\boxed{2'}} + x^{\boxed{1'}\boxed{2'}} \\ &= 2x_1^2 + 2x_2^2 + 4x_2x_1 \text{ (descending)} \\ &= 2x_1^2 + 2x_2^2 + 2x_2x_1 + 2x_1x_2 \text{ (hook)} \\ &= \frac{1}{2}Q_{(1)}^2 \end{aligned}$$

Hook reading is the more natural.

Proposed Rule of Three

Conjecture

Let $u_1, \dots, u_N, v_1, \dots, v_N$ be elements of a ring A. The following are equivalent:

- $Q_{(k)}(u_S)$ *and* $Q_{(\ell)}(v_S)$ *commute for all* S, k, ℓ.
- *the above holds when* $k = 1$ *or* $\ell = 1$ *(for all* S*)*

Computation suggests this is optimal.

Naive Approach

$$aba + bba - baa - bab$$
$$aca + cca - caa - cac$$
$$bcb + ccb - cbb - cbc$$
$$acb + bca - cab - bac$$
$$acba + caba - cbca - cbac$$

Can we get the next simplest commutation relation?

$$\begin{aligned} C &= [e_2(a,b,c), e_3(a,b,c)] \\ &= (cb + ca + ba)(cba) - (cba)(cb + ca + ba) \end{aligned}$$

Yes!

$$\begin{aligned} C = {} & (bcb + ccb - cbb - cbc)(aa + ab - ba) \\ & + (cc + ac + bc - cb - ca)(aba + bba - baa - bab) \\ & - (aca + cca - caa - cac)ba \\ & - (acb + bca - cab - bac)ba \\ & + (acba + caba - cbca - cbac)(a + b) \end{aligned}$$

Generating Functions

In the *commutative* case:

$$\begin{aligned} a_i &= 1 + xu_i \\ b_i &= (1 - xu_i)^{-1} \\ q_i &= a_i b_i = (1 + xu_i)(1 - xu_i)^{-1} \end{aligned}$$

And:

$$\begin{aligned} a_{[n]} &= \sum e_k x^k \\ b_{[n]} &= \sum h_k x^k \\ q_{[n]} &= \sum Q_{(k)} x^k \end{aligned}$$

Generating Functions (cont.)

In the *non-commutative* case:

$$\begin{aligned} a_i &= 1 + xu_i \\ b_i &= (1 - xu_i)^{-1} \\ q_i &= a_i b_i = (1 + xu_i)(1 - xu_i)^{-1} \end{aligned}$$

And:

$$\begin{aligned} a^{\downarrow}_{[n]} &= \sum e_k x^k \\ b^{\uparrow}_{[n]} &= \sum h_k x^k \\ q^{\downarrow}_{[n]} &= \sum Q_{(k)} x^k \text{ descending reading} \end{aligned}$$

Generating Functions (cont.)

In the *non-commutative* case:

$$a_i = 1 + xu_i$$
$$b_i = (1 - xu_i)^{-1}$$

And:

$$a^{\downarrow}_{[n]} = \sum e_k x^k$$
$$b^{\uparrow}_{[n]} = \sum h_k x^k$$
$$a^{\downarrow}_{[n]} b^{\uparrow}_{[n]} = \sum Q_{(k)} x^k \text{ hook reading}$$

Rephrasing Rule of Three

Theorem (Blasiak-Fomin 3.5)

Let R be a ring, and let $g_1, \ldots, g_N, h_1, \ldots, h_N \in R$ be potentially invertible elements. Then the following are equivalent:

- $\left[\sum_{i \in S} g_i, \sum_{i \in S} h_i\right] = 0$,
  $\left[\sum_{i \in S} g_i, h_S\right] = 0$,
  $\left[g_S, \sum_{i \in S} h_i\right] = 0$,
  $[g_S, h_S] = 0$ *for all subsets* S.
- *the above holds for* $|S| \leq 3$.

Use $g_i = 1 + xu_i$, $h_i = 1 + yv_i$ to get rule of three for e_ks.

Conjecture

Let A be a ring, and let $\{x_i\}_{i\in[N]}, \{y_i\}_{i\in[N]} \in A$. Then define

$$a_i = 1 + x_i t \qquad b_i = 1 - x_i t$$
$$\alpha_i = 1 + y_i s \qquad \beta_i = 1 - y_i s$$

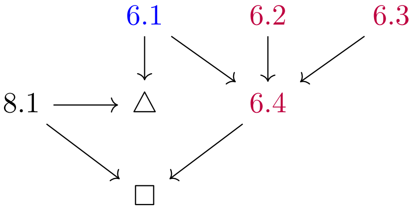
Further, let the following be true.

$$\left[\sum_{i\in S} x_i, \alpha_S(\beta_S)^{-1}\right] = 0$$
$$\left[\sum_{i\in S} y_i, a_S(b_S)^{-1}\right] = 0$$

Then $a_{[N]}(b_{[N]})^{-1}\alpha_{[N]}(\beta_{[N]})^{-1} = \alpha_{[N]}(\beta_{[N]})^{-1}a_{[N]}(b_{[N]})^{-1}$.

Proof Progress

We have been attempting to replicate Blasiak and Fomin's proof of Lemma 8.2, both in the standard case, and in the weakened case where nonadjacent variables commute.



□: proof for the conjecture;

△: proof for the conjecture for $|S| = 2$.

Arrows represent dependencies.

Further Questions

- Can we prove the conjecture? What about in a weaker setting?
- When does commutativity extend to all Schur Q-functions?

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