We'll work over a field $k$, and think $k = \mathbb{C}$ so that polynomials have roots.

$$\mathbb{C}^* = \mathbb{C}^* = \mathbb{C} - \{0\}$$

a (1-dimensional) algebraic torus
Given \( \bar{x} = (x_0, x_1, \ldots, x_n) \in \mathbb{C}^{n+1} \), consider
\[
\begin{aligned}
\mathbb{C}^x \times \mathbb{C}^{n+1} & \rightarrow \mathbb{C}^{n+1} \\
(t, \bar{x}) & \rightarrow (tx_0, tx_1, \ldots, tx_n) \\
& = t \bar{x}
\end{aligned}
\]

For \( \bar{x} \in \mathbb{C}^{n+1} \setminus \{0\} \), write
\[
[\bar{x}] = \{t \bar{x} \mid t \in \mathbb{C}^x\}
\]

The \textit{corbit} of \( \bar{x} \) \( \leftrightarrow \) a line thru \( 0 \) in \( \mathbb{C}^{n+1} \).

\textit{Projective space} \( P^n = \{[\bar{x}] \mid \bar{x} \in \mathbb{C}^{n+1} \setminus \{0\}\} \)

Note \( \mathbb{C}^{n+1} \setminus \{0\} \rightarrow P^n \)
\[
\begin{aligned}
\bar{x} & \rightarrow [\bar{x}] \\
\text{and} \quad [\bar{x}] = [\bar{y}] & \iff \exists t \in \mathbb{C}^x \text{ with } t\bar{x} = \bar{y}
\end{aligned}
\]
For \( \alpha \in \mathbb{N}^{n+1} \), let \( x^\alpha = x_0 x_1 \cdots x_n \) and \( |\alpha| = \sum_{i=0}^{n} \alpha_i \)

If \( f(x) = \sum_{\alpha} a_\alpha x^\alpha \in \mathbb{C}[x_0, x_1, \ldots, x_n] \) polyomials

\[ f(t \cdot x^\alpha) = \sum_{\alpha} a_\alpha (t \cdot x^\alpha) = \sum_{\alpha} a_\alpha t^{|\alpha|} x^\alpha. \]

**DEFN:** Say \( f(x) \) is homogeneous (of degree \( d \)) if \( \exists d \) with \( |\alpha| = d \) for \( \alpha \) above with \( a_\alpha \neq 0 \).
Note: \( f(\bar{y}) = 0 \ \forall \bar{y} \in [\bar{x}] \)
follows from \( f(x) = 0 \)

\( \iff \) \( f(x) \) is homogeneous

Given \( X \subseteq \mathbb{P}^n \), set

\[
I(X) := \langle \{ f(x) \in \mathcal{S} \mid f(\bar{c}) = 0 \ \forall \bar{c} \in X \} \rangle
\]

\( C[\bar{x}_0, \ldots, \bar{x}_n] \)

(later: \( I(X) \) is always generated by homogeneous polynomials)
EXAMPLES:

1. $P^2 \subset X = \{ [1:0:0], [0:1:0], [0:0:1] \}$

   $[1:0:0] = C^y(1,0,0) = \{ (t,0,0) : t \in C \}$

Here,

   \[ I(X) = \langle x_1x_2 \rangle \cap \langle x_0x_2 \rangle \cap \langle x_0x_1 \rangle \]

   \[ = \langle x_0x_1, x_1x_2, x_0x_2 \rangle \]

2. $P^2 \supset Y = \{ [1:0:0], [1:1:0], [2:0:1] \}$

   \[ I(Y) = \langle x_1x_2 \rangle \cap \langle x_0x_1x_2 \rangle \cap \langle x_0x_2, x_1 \rangle \]

   \[ = \langle x_1x_2, x_0x_2 - 2x_2^2, x_0x_2 - x_2^3 \rangle \]
How'd we compute those ideal intersections?

Crash course in algebraic geometry

**DEFN**: $I \subseteq S = \mathbb{C}[x_0, x_1, \ldots, x_n]$ is an ideal if

1. $0 \in I$ (or $I \neq \emptyset$)
2. $a, b \in I \Rightarrow a + b \in I$
3. $a \in I, \; f \in S \Rightarrow af \in I$

**Claim**: $I(\mathbb{C})$ above is always an ideal
Given \( f_1, \ldots, f_r \in S \), the ideal generated by \( f_1, \ldots, f_r \) is

\[
(\mathfrak{a}) \langle f_1, \ldots, f_r \rangle = \left\{ \sum_{i=1}^r h_i f_i : h_i \in S \right\}
\]

e.g.

\[
\langle x_0 x_1, x_0 x_2, x_1 x_2 \rangle = \left\{ a(x) x_0 x_1 + b(x) x_0 x_2 + c(x) x_1 x_2 \right\}
\]

---

**Hilbert's Basis Theorem**

S is a Noetherian ring, meaning every ideal \( I \subset S \) is finitely generated, that is, of the form \( I = \langle f_1, \ldots, f_r \rangle \).
**DEF'N**: An ideal $I \subset S$ is **homogeneous** if it can be generated by homogeneous polynomials.

**CLAIM**: $X \subset \mathbb{P}^n \Rightarrow I(X)$ homogeneous

**DEF'N**: If $I = \langle f_1, \ldots, f_r \rangle$ is a homogeneous ideal in $S$, then

$V(I) = \{ p \in \mathbb{P}^n \mid f_1(p) = \ldots = f_r(p) = 0 \} = \{ p \in \mathbb{P}^n \mid f(p) = 0 \ \forall f \in I \}$

is a **projective algebraic variety**.
EXAMPLE:
\[ V(\langle x_0 x_1, x_0 x_2, x_1 x_2 \rangle) = \{ [1:0:0], [0:1:0], [0:0:1] \} \subset \mathbb{P}^2 \]

THE GAME:
Geometric properties of \( V(I) \)

\[ \leftrightarrow \]
Algebraic properties of the ring \( \mathbb{C}[x_0, \ldots, x_n]/I = S/I \).

e.g. irreducible varieties

\[ \leftrightarrow \] domains \( S/I \) (i.e. \( I \) a prime ideal)
Recall: \( X \subset \mathbb{P}^n \)
\[ \Rightarrow I(X) = \{ f \in S \mid f(p) = 0 \text{ for } p \in X \} \]

**Theorem:** For \( k \) infinite, the maps \( I \) are inclusion-reversing.

Furthermore, for any proj. variety \( V \),
\[ V(I(V)) = V. \]

In other words,
\( X \subsetneq Y \Rightarrow I(X) \supsetneq I(Y) \)
\( I \subsetneq J \Rightarrow V(I) \supsetneq V(J) \).
**Example:** \( V(x_0) = V(x_0^2) \),

so \( I(V(I)) \supseteq I \)

(e.g. take \( I = \langle x_0^2 \rangle \))

**How to fix this?**

**Defn:** If \( I \) is an ideal of \( S \), then the **radical** of \( I \) is

\[
\sqrt{I} = \{ f \in S : \exists n \in \mathbb{Z}_{>0} \text{ with } f^n \in S \}
\]

**Examples:**

\[
\sqrt{\langle x_0^2 \rangle} = \langle x_0 \rangle
\]

\[
\sqrt{\langle x_0^2 y^3 \rangle} = \langle x_0 y \rangle
\]

Need binomial theorem.
THEOREM
(Projective strong Nullstellensatz)

If \( k \) is algebraically closed, \( I \) a homogeneous ideal in \( S = \mathbb{k}[x_0, \ldots, x_n] \) and \( V(I) = \emptyset \) is a nonempty projective variety in \( \mathbb{P}^n \), then
\[
I(V(I)) = \sqrt{I}.
\]
Projective Ideal-Variety Correspondence

If we restrict the earlier correspondence, we get

\[
\begin{align*}
\{ \text{nonempty projective varieties} \} & \quad \overset{I}{\leftrightarrow} \quad \{ \text{radical homogeneous ideals} \} \\
\lor & \quad \lor & \quad \lor
\end{align*}
\]

as inclusion-reversing mutually inverse bijections.

Note: Primary decomposition of ideals explains how to write varieties down as unions of irreducible varieties.
a) Prove that (*) is an ideal.

b) If $I \subset S$ is a (homogeneous) ideal, show that $\sqrt{I}$ is (homogeneous) ideal.

c) Let $f, g \in \mathbb{C}[x, y]$ be distinct nonconstant polynomials. Let $I = \langle f^2, g^3 \rangle$. Is it true that $\sqrt{I} = \langle f, g \rangle$? Explain.

d) Let $I, J$ be homog. ideals in $S$. Show $V(INJ) = V(I) \cup V(J)$. 
Hilbert functions

$S_d = \{\text{homog. polynomials of degree $d$}\}$

a $\mathbb{C}$-vector space

$S = \mathbb{C}[x] = \bigoplus_{d=0}^{\infty} S_d$

$\dim_{\mathbb{C}} S_d = \#\{\text{monomials of degree $d$ in } \mathbb{C}[x_0, \ldots, x_n]\}$

$= \binom{n+d}{n} = \frac{(n+d)!}{n! \cdot d!}$

Why? $m=2, d=5$

$x_0^3x_2^2 \prec \ldots \prec (0,0,0,2,2)$

$\{0+1, 0+2, 0+3, 2+4, 2+5\}$

$= \{1,2,3,6,7\} \subset \{1,2,\ldots, m+d\}$
Let's consider the function

\[ \text{HF}_{s/I} : \mathbb{Z} \longrightarrow \mathbb{N} \]

\[ d \longrightarrow \dim_c(S/I)_d \]

if \( I \) is homogeneous.

**EXAMPLE:** \( S = I(x) = \langle x_0 x_1, x_0 x_2, x_0 x_2 \rangle \)

<table>
<thead>
<tr>
<th>( d )</th>
<th>monomials of degree ( d ) in ( S )</th>
<th>( \dim_c(S/I)_d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( x_0, x_1, x_2 )</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>( x_0^2, x_1 x_2, x_2^2, x_0 x_0 x_1, x_0 x_1 x_2 )</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( d )</td>
<td>( x_0^d, x_1^d, x_2^d )</td>
<td>3</td>
</tr>
</tbody>
</table>
**Example**: \( \Sigma_{I}(C) = \langle x_0 x_2 - x_1^2 \rangle \)

<table>
<thead>
<tr>
<th>( d )</th>
<th>monomials</th>
<th>( \dim \mathcal{E}(\Sigma_{I})_d )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>( x_0, x_1, x_2 )</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>( x_0^2, x_1^2, x_2^2, x_0 x_1, x_1 x_2, x_0 x_2 )</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>(check ( \rightarrow ))</td>
<td>7</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( d )</td>
<td>( 2d + 1 )</td>
<td>( \vdots )</td>
</tr>
</tbody>
</table>

\( C \) is a curve (1-dimensional in \( \mathbb{P}^2 \)) and \( H_{\Sigma_I}(d) \) had degree 1 as a polynomial in \( d \).
Hilbert's polynomial theorem

Given homogeneous I ⊆ S,

exists a polynomial \( P(z) \in \mathbb{Q}[z] \)
such that, for \( d \) sufficiently large,

\[
HF_{S/I}(d) := \dim_\mathbb{Q}(S/I)_d = P(d).
\]

(The polynomial \( P(z) \) is called the Hilbert polynomial of \( I \).)

We'll deduce this from free resolutions of \( S/I \).
EXAMPLE: $I = I(x) = \langle x_1 x_2, x_1 x_2, x_1 x_2 \rangle$

$O \leftarrow S/I \leftarrow \begin{bmatrix} \chi_2 & \chi_2 \\ -\chi_1 & 0 \\ 0 & -\chi_1 \end{bmatrix}$

$S \leftarrow S^3(-2) \leftarrow S^2(-3) \leftarrow 0$

$\begin{align*}
&= \dim_c(S/I)_d = \dim_c S_d - 3 \dim_c S^2_d \\
&\quad + 2 \dim_c S^3_d \\
&= \dim_c S_d - 3 \dim_c S_{d-2} + 2 \dim_c S_{d-3}
\end{align*}$

$= 3 \text{ for } d \gg 0$

\[\begin{align*}
&= \dim_c S_d - 3 \dim_c S_{d-2} + 2 \dim_c S_{d-3} \\
&= 3 \text{ for } d \gg 0
\end{align*}\]
If $m+d-a \geq 0$, then
\[
\dim C_{d-a} = \binom{m+d-a}{m}
= \frac{(m+d-a)(m+d-a-1)\ldots(d-a+1)}{m!}
\]

\underline{Hilbert's Syzygy Theorem}\quad \text{on } \mathbb{C}P^m,

$I(X) \subset S = \mathbb{C}[x_0, \ldots, x_m]$ always has a free resolution
\[
0 \leftarrow S/I \leftarrow S \leftarrow F_1 \leftarrow F_2 \leftarrow \ldots \leftarrow F_m \leftarrow 0
\]

of length at most $m$. 
REU Exercice 12

a) $I = \langle x_0 x_1 \rangle \cap \langle x_2 x_3 \rangle$
(2 skew lines in $\mathbb{P}^3$)

Prove $I = \langle x_0 x_2, x_1 x_3, x_0 x_3, x_1 x_2 \rangle$

Compute the Hilbert function polynomial
free resolution
(find one of length 3, show this is the minimal length)
b) Show $V(J) = V(I)$ for 
\[ J = \langle x_0 x_2 - x_1 x_3, x_0 x_3, x_1 x_2 \rangle, \]
but $JC \neq I(V(I))$.

Compute the Hilbert function polynomial free resolution.

Hint: Show 
\[ J = I \cap \langle x_3^2, x_0 x_3, x_2^2, x_1 x_2, x_0 x_2 - x_1 x_3, x_1 x_2 - x_1 x_3 \rangle \]

C) $R = k[x]/\langle x^3 \rangle$

Compute a free resolution of $R/\langle x^3 \rangle$ as an $R$-module, not $S$-module. In particular, show it is infinite.
Virtual resolutions

\[ \mathbb{P}^{\overline{n}} := \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \ldots \times \mathbb{P}^{n_r} \]

\[ ((t_1, \ldots, t_r), (x_1, x_2, \ldots, x_r)) \mapsto (t_1 x_1, \ldots, t_r x_r) \]

Set \( \deg(x_i) = i^{th} \) standard basis vector in \( \mathbb{Z}^r \)

\[ S = \mathbb{C}[x_1, \ldots, x_r] \text{ is a } \mathbb{Z}^r \text{-graded ring} \]
EXAMPLE: \( \mathbb{P}^1 \times \mathbb{P}^2 \): 

\[
\begin{align*}
\left( \mathbb{C}^* \right)^2 \times \mathbb{C}^5 & \rightarrow \mathbb{C}^5 \\
(t_0, t_2, (x_0, x_1, y_0, y_1, y_2)) & \mapsto (tx_0, tx_1, \\
& \quad ty_0, ty_1, ty_2)
\end{align*}
\]

\[
\begin{align*}
\deg(x_0) &= \deg(x_1) = 1 \\
\deg(y_0) &= \deg(y_1) = \deg(y_2) = 1 \\
\deg(x_0^2 x_1 y_0^5) &= 3
\end{align*}
\]

Instead of throwing away \( \{0\} \times \mathbb{C}^{n+1} \) for \( \mathbb{P}^{n+1} \), here we throw away \( \{0\} \times \mathbb{C}^3 \cup (\mathbb{C}^* \times \{0\}) \).

\[ B = \langle x_0, x_1 \rangle \cap \langle y_0, y_1, y_2 \rangle \]
Geometrically, $V(B) = \emptyset < \mathbb{P}^1 \times \mathbb{P}^2$.

**DEFN:** A virtual resolution of $S/I$ is a sequence of $F_i = \bigoplus (S(-\alpha))\beta_i \alpha$ such that $0 \leftarrow F_0 \leftarrow F_1 \leftarrow \cdots \leftarrow F_t \leftarrow 0$ has $\text{ann} \left( \frac{\ker \partial_i}{\text{im} \partial_i+1} \right) \supseteq B^l$ for $l \gg 0$ and $\text{ann} \left( \frac{F_0}{\text{im} \partial_0} \right) = I$, up to components of $B$.

See examples in Macaulay 2 demo to eventually appear.
Virtual Hilbert Syzygy Theorem

$\exists Y \subset \mathbb{P}^n$, $I(Y)$ has a virtual resolution of length

$\leq |\bar{n}| = n_1 + n_2 + \ldots + n_r$.

Points in $\mathbb{P}^1 \times \mathbb{P}^1$

\[\begin{array}{cccc}
4 & \bullet & \bullet & \bullet \\
3 & \bullet & \bullet & \bullet \\
2 & \bullet & \bullet & \bullet \\
1 & \bullet & \bullet & \bullet \\
\end{array}\]

$\mathbb{P}^1 \rightarrow \mathbb{P}^1$
REU Exercise 13

a) What are all possible configurations of 3 points in $\text{P}^1 \times \text{P}^1$?

b) Write out their defining ideals.

c) Compute their corresponding Hilbert functions $HF$ polynomials $HP$

free resolutions

d) Compute virtual resolutions in each case (get length 2).

e) Do same for 4 points

f) Write Macaulay2 code to compute $I(R)$
REU Problem 5

For configurations of points in $P^1 \times P^1$ (later $P^a \times P^b$),

a) What powers of components of $B$ give a "short" virtual resolution.

b) What is the minimal number of generators needed to generate an ideal of points virtually?

c) When does the ideal of points have a virtual resolution that is a Koszul complex?

d) More to do!
... switched to Macaulay2
demo of virtual resolutions.

Starting with free resolution, can intersect with powers of components of $B$, and sometimes this gives virtual resolutions of $I(Y)$, smaller than the original.