1. Reflection & Coxeter groups
2. Weight polytopes
3. Simple polytopes & F-vectors
4. REU problem 1.7

DEF: A reflection t acting on V = \mathbb{R}^n is an element t \in GL_n(\mathbb{R}) that fixes a hyperplane H and interchanges the line H^⊥ (perpendicular to H).

H is called the reflection hyperplane for t.

EX1: K = 5

W = I_5(k) for k ≥ 3

= "dihedral group of order 2k"

= linear symmetries of a regular k-sided polygon P

(= \{ g \in GL_n(\mathbb{R}) : g(P) = P \})

EX2: K = 4

I_4

REU Exercise 13:
(a) Prove I_3(k) \cong \{ e, s, s^2, \ldots, s^{k-1} \} \cong \{ 0, s, 2s, \ldots, (k-1)s \} 

GL_n(\mathbb{R}) rotations reflections

(b) Prove the abstract presentation I_3(k) \cong \langle s, t : s^k, t^2, ts = st \rangle

(c) Prove the Coxeter presentation I_3(k) \cong \langle s, t : s^k, t^2, (st)^2 = st \rangle

Hint: create well-defined maps between presentations satisfying the necessary relations, i.e., for (c), where should you map s, t?

(c) gives you sufficiency for your argument

DEF: A Coxeter presentation for a group W is of the form

W \cong \langle s_1, \ldots, s_n \rangle \langle s_i^2 = 1, (s_i s_j)^{m_{ij}} = 1 \rangle

for some m_{ij} \in \{ 2, 3, 4, \ldots \} u \mathbb{N}

EX1: S_4 acts by reflection symmetries.
The hyperplanes/chambers of $\mathbb{S}_n$ cut out the Coxeter complex on the boundary of the tetrahedron.

**Example 1.**

$W = \mathbb{S}_4$

$S = \{ e, s_1, s_2, s_3 \}$


gives rise to the Coxeter diagram:

![Coxeter diagram](image)

which has Coxeter diagram of type $A_2$.

More generally, for $W = \mathbb{S}_n$, $S = \{ e, s_1, s_2, \ldots, s_{n-1} \}$

which has Coxeter diagram of type $A_{n-1}$.

Some finite reflection groups $W$ acting on $V = \mathbb{R}^n$ stabilize a full rank lattice $\Lambda \subseteq \mathbb{Z}^n$ inside $\mathbb{R}^n$, and are called crystallographic reflection groups, or Weyl groups.

**Example 2.**

$V = \mathbb{R}^2$

$S_4 = \{ e, s_1, s_2, s_3 \} = \mathbb{S}_3$.

$I_3(4) = B_3$.

$I_3(5) = B_3$.

REU Exercise #18:

Show a Coxeter system $(W, S)$ with $W$ crystallographic must have all $W = \{ e, s_2, s_3, s_4, s_5, s_6 \}$.

Remark: IF $(W, S)$ has $W$ finite and all $W = \{ e, s_2, s_3, s_4, s_5, s_6 \}$, then $W$ is a Weyl group.

Remark: Weyl groups have associated (linear) algebraic groups, like $G = GL_n(F)$ for $W = \mathbb{S}_n$, with Borel subgroups $B$ (upper $\Delta$ for $GL_n$) and Bruhat decompositions $G = \cup BwB$.

$W$ "controls" the representation theory and structure of $G$.

2. Weight polytopes ($\pm$ Wythoff's construction)

**Definition**: Given $W$ a finite reflection group acting on $V = \mathbb{R}^n$, and pick a $\lambda \in V$, then $P_\lambda$ is the weight polytope for $\lambda$.

Find the convex hull of the $W$-orbit of $\lambda$.

The smallest cone containing the $W(\lambda)$.

**Example 3.**

$W = \mathbb{S}_5$

$I_3(4) = (5, 4)$.

**Example 4.**

$W = \mathbb{S}_4$

$I_3(4) = (5, 4)$.

3. Simple polytopes $P$ and $\mathbb{S}_n$:

**Definition**: For a convex polytope $P$ of dimension $n$, its $f$-vector $f(P) = (f_0, f_1, \ldots, f_n)$ where $f_i$ is the number of $i$-dimensional faces of $P$.

**Example**: Our first choice of $\lambda$ for $W = \mathbb{S}_4$ gave us $P_\lambda$ with $f(P) = (2, 10, 10, 1)$.
Since the W-orbit of \( F_2 \) in \( P_2 \) (for \( \pi \neq e(A) \)) looks like cosets \( W/W_{2} \) where \( W_{2} \) is the \( W \) stabilizer of \( F_2 \) and has size \( 1/W/W_{2}=1/W/W_{1}=[W/W_{0}^{2}]_{1} \), \( \dim(F_2)=111 \).

Given: \( P(A) = p \sum_{(W_{2})} \frac{1}{W/W_{2}} \frac{1}{W/W_{1}} \frac{1}{W/W_{0}^{2}} \)

**Type B/C:**

![Type B/C Diagram](image)

Frieden's results from the Weyl group hypothesis, by showing they were already known to Maxwell, Scharlau...

(b) Free Renner's results from the Weyl group hypothesis, by showing they were already known to Maxwell, Scharlau...

For \( P \) simple,
- \( h(P) \) is always symmetric \( (\eta_{i} = \eta_{n-i}) \)
- \( h_{i} \geq 0 \)
- \( h_{i} \) have various algebraic & topological interpretations

**EXIII** \( W = S_{n} \) and \( \lambda \) generic

**EXIV** \( e^{-P_{2}} \) is called a permutahedron

\[ h(P, \lambda) = E_{\lambda}(t) \] is called the Eulerian polynomial

\[ E_{\lambda}(t) = \sum_{\omega \in \lambda} t^{|\omega|} \]

The \( E_{\lambda}(t) \) compile nicely in an exponential generating function (EGF),

\[ \sum_{\lambda \in \lambda} E_{\lambda}(t) \frac{m!}{m!} = \frac{e^{x \lambda(n)}}{x} \]

### RELU Problem #71

(a) Use Renner's classification of the simple \( P \)'s in all types [Renner 2009, Thm 3.1] and continue the work of Gashnikov (2008) by computing the \( \eta/\eta \) vectors (and compiling them in generating functions) for them as families.

e.g., Type A

![Type A Diagram](image)