SEALow Team 2 Presents:
More Miles for Your (Sand)Piles

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Overview

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Definitions and Previous Results

Section 1
Definitions

Definition

Given $F_2^r, M$, where $M = \{v_1, \ldots, v_n\}$ is a set of generators, we define the Cayley graph $G(F_2^r, M)$ with $V(G) = F_2^r$ and $u, w \in V(G)$ share an edge if $u - w = v_i$ for some generator. Multiple edges are allowed.

Example

- Let $M = \{e_1, \ldots, e_n\}$. Then $G(F_2^r, M) = Q_n$, is called the hypercube graph.
- If $M = \{v \in F_2^r - \{0\}\}$, then $G(F_2^r, M) = K_{2^r}$ is called the complete graph on $2^r$ vertices.
- See board for image of $Q_2$ and $K_4$ with generators labelled.
Definitions

Definition

The Laplacian of a nondirected graph \( G \), denoted \( L(G) \) has entries

\[
L(G)_{i,j} = \begin{cases} 
    \text{deg}(v_i) & i = j \\
    -\#\text{edges from } v_i \text{ to } v_j & i \neq j
\end{cases}
\]

Definition

Given a connected graph \( G \) with \( |V(G)| = w \), \( L(G) \) is an integer \( w \times w \) matrix, so we can view it as map of \( \mathbb{Z} \)-modules \( \mathbb{Z}^w \to \mathbb{Z}^w \). The kernel is \( \text{span}(1) \), so \( \text{coker } L(G) \cong \mathbb{Z} \oplus K(G) \) where \( K(G) \) is a finite abelian group. We call \( K(G) \) the \textbf{sandpile group} of \( G \).

Example

It is well known that \( K(K_n) \cong (\mathbb{Z}/n\mathbb{Z})^{n-2} \). So we can determine at least one case of Cayley graphs.
Definitions and Previous Results

Previous Results for $\mathbb{F}_2^r$

Lemma

Let $M = \{v_1, \ldots, v_n\}$ be the set of generators. For every $u \in \mathbb{F}_2^r$, let

$$f_u = \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} e_v$$

$$\lambda_{u,M} = n - \sum_{i=1}^{n} (-1)^{u \cdot v_i}$$

Then $\{f_u\}$ is an eigenbasis of $\mathbb{R}^{2^r}$ each with eigenvalues $\{\lambda_{u,M}\}$, which is always even. Moreover, $e_v = \frac{1}{2^r} \sum_{v \in \mathbb{F}_2^r} (-1)^{u \cdot v} f_v$.

Theorem (Ducey-Jalil)

Let $G$ be a Cayley graph of $\mathbb{F}_2^r$. For all $p \neq 2$,

$$\text{Syl}_p(K(G)) \cong \text{Syl}_p \left( \bigoplus_{k=1}^{2^r} \mathbb{Z}/\lambda_{u,M} \mathbb{Z} \right)$$
Remark

$L(G)$ is diagonalizable over $\mathbb{Z}[\frac{1}{2}]$, and we can describe the Sylow-$p$ structure for all $p \neq 2$ in terms of the eigenvalues.

What about $p = 2$? Is the Sylow-2 group uniquely determined by the eigenvalues?

Theorem

There is an isomorphism of abelian groups

$$\mathbb{Z} \oplus K(G) \cong \mathbb{Z}[x_1, \ldots, x_r]/\left( x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{i=1}^{n} \prod_{j} x_j^{(v_i)_j} \right)$$
Previous results for $p = 2$

**Theorem (Bai)**

For $G = Q_n$, the number of Sylow-2 cyclic factors is $2^{n-1} - 1$. Additionally, the number of $(\mathbb{Z}/2\mathbb{Z})$’s in $K(G)$ is $2^{n-2} - 2^{\lfloor (n-2)/2 \rfloor}$.

**Theorem (Anzis-Prasad)**

The size of the largest factor in $Syl_2(K(Q_n))$ is $\leq 2^{n+\lceil \log_2 n \rceil}$.

We will generalize Bai’s first result and Anzis-Prasad, but not Bai’s second result.
Section 2

Results on the Number of Even Invariant Factors
Invariant factors

**Definition**

We define $d(M)$ to be the number of Sylow-2 cyclic factors in $K(G)$.

**Proposition (Parity Invariance)**

Let our matrix of generators $M$ have multiplicities $(a_{v_1}, \ldots, a_{v_{2^r-1}})$ for each nonzero vector in $\mathbb{F}_2^r$. Then $d(M)$ only depends on the parity of the multiplicities of generators.

**Example**

If $M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $M' = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}$ then

$$K(G(\mathbb{F}_2^3, M)) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z})$$

$$K(G(\mathbb{F}_2^3, M')) = (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z}) \oplus (\mathbb{Z}/120\mathbb{Z})$$
Computing $d(M)$

Definition
A Cayley graph $G(F_2^r, M)$ with $M = \{v_1, \ldots, v_n\}$ is called \textbf{generic} if $\sum_{i=1}^n v_i \neq \vec{0}$. For example, $Q_n$ is generic for all $n \geq 1$.

Theorem
If $G(F_2^r, M)$ is generic, then $d(M) = 2^{r-1} - 1$.

Proof Sketch.
Consider $(\mathbb{Z} \oplus K(G)) \otimes (\mathbb{Z}/2\mathbb{Z})$. $d(M)$ is equal to the dimension of $K(G) \otimes (\mathbb{Z}/2\mathbb{Z})$ as a vector space. Theorem’s condition gives us a nonzero degree 1 term of the form $u_i$ which allows us to construct an explicit isomorphism $(\mathbb{Z} \oplus K(G)) \otimes (\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^{2^{r-1}}$. □
Conjectures

Conjecture

For a collection of generators, $M$, yielding a connected Cayley graph on $\mathbb{F}_2^r$, $d(M) \geq 2^{r-1} - 1$ with equality occurring iff $M$ is generic.

Conjecture

$d(M)$ is odd unless all of the eigenvalues have the same power of 2, in which case $d(M) = 2^n - 2$. 
Reducing the Sandpile Group and Results for Small Cases
Reducing multiplicities in $M$

Given generators for $V = \mathbb{F}_2^r$, we can express these generators in terms of their multiplicities $\vec{a} = (a_{v_1}, \ldots, a_{v_{2^r-1}})$, where the multiplicity, $a_{v_i}$, denotes the number of times the vector $v_i$ occurs. Here, we will use the binary naming convention for vectors, so $v_3 = (1, 1, 0)$.

**Lemma**

Let $G_1 = G(\mathbb{F}_2^r, M_1)$ and $G_2 = G(\mathbb{F}_2^r, M_2)$ such that $\vec{a}_2 = \lambda \vec{a}_2$ for $\lambda \in \mathbb{N}$ and let $\{\alpha_i\}$ be the invariant factors in the Smith Normal Form of $L(G_1)$. Then

\[
K(G_1) = \prod_{i=1}^{2^r} \mathbb{Z}/\alpha_i \mathbb{Z} \implies K(G_2) = \prod_{i=1}^{2^r} \mathbb{Z}/(\lambda \alpha_i) \mathbb{Z}
\]

**Proof.**

$\vec{a}_2 = \lambda \vec{a}_2 \implies L(G_2) = (\lambda \text{Id}) \cdot L(G_1)$. Now consider SNF of $L(G_2)$. (Note: reduces analysis to $gcd(\vec{a}) = 1$ case.)
Example

Consider the two matrices of generators:

\[ M_1 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \]
\[ \Rightarrow K(G(\mathbb{F}_2^2, M_1)) = (\mathbb{Z}/1\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \oplus (\mathbb{Z}/4\mathbb{Z}) \]

\[ M_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix} \]
\[ \Rightarrow K(G(\mathbb{F}_2^2, M_2)) = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z}) \]
Theorem

Given a matrix of generators on $\mathbb{F}_2^r$

\[
M = \begin{pmatrix}
v_1 & v_2 & \cdots & v_n \\
\end{pmatrix}
\]

and an element $g \in \text{GL}_r(\mathbb{F}_2)$, define $M' := g \cdot M$, then $G(\mathbb{F}_2^r, M)$ and $G(\mathbb{F}_2^r, M')$ have the same sandpile group.

Proof.

An element of $\text{GL}_r$ permutes the nonzero vertices of the graphs and the edges in a consistent manner. This induces a graph isomorphism, and thus a sandpile group isomorphism.
Example

Assume $a_{v_1} = a_{v_2} = a_{v_4} = a_{v_8} = 2$. Let $\{i_1, \ldots, i_k\}$ denote $a_{v_{i_1}} = \cdots = a_{v_{i_k}} = 2$ and $a_{v_j} = 1$ for all $j \notin \{1, 2, 4, 8\} \cup \{i_1, \ldots, i_k\}$:

$$\{6, 10, 12\}, \{5, 9, 12\}, \{3, 5, 6\}, \{3, 9, 10\}, \{10, 12, 14\},$$
$$\{9, 12, 13\}, \{5, 6, 7\}, \{3, 10, 11\}, \{6, 12, 14\}, \{5, 9, 13\}, \{5, 12, 13\},$$
$$\{3, 6, 7\}, \{3, 9, 11\}, \{6, 10, 14\}, \{3, 5, 7\}, \{9, 10, 11\}$$

All of the 16 previous cases yield

$$K(G) = (\mathbb{Z}/3\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/48\mathbb{Z}) \oplus (\mathbb{Z}/48\mathbb{Z})$$
$$\oplus (\mathbb{Z}/528\mathbb{Z}) \oplus (\mathbb{Z}/6864\mathbb{Z}) \oplus (\mathbb{Z}/6864\mathbb{Z})$$
Size of 2-Sylow component

By Kirchoff’s Matrix Tree Theorem

$$|K(G)| = \det L(G)^i,i = \frac{\lambda_2 \cdots \lambda_m}{m}$$

where $\lambda_1 = 0$ is only 0 eigenvalue by convention. Here, $m = 2^r$, so

$$|Syl_2(K(G))| = \frac{1}{2^r} \text{Pow}_2 \left( \prod_{u \in \mathbb{F}_2^r \setminus \{0\}} \lambda_{u,M} \right)$$

where for $n = 2^k \cdot b$ with $k$-maximal, we define $\text{Pow}_2(n) := 2^k$ and $v_2(n) := k$. 
In the generic case for $r = 2$ with multiplicities $\vec{a} = (a_1, a_2, a_3)$, we have $\text{Syl}_2(K(G)) \cong \mathbb{Z}/2^e\mathbb{Z}$ with

$$\lambda_2 = 2(a_1 + a_3), \quad \lambda_3 = 2(a_2 + a_3), \quad \lambda_4 = 2(a_1 + a_2)$$

$$\vec{a} \equiv (1, 0, 0) \implies e = v_2(\lambda_2\lambda_3\lambda_4) - 2 = v_2(a_2 + a_3) + 1$$

$$\vec{a} \equiv (1, 1, 0) \implies e = v_2(a_2 + a_3) + 1$$

by $GL$ equivalence of generators, these handle all generic cases. Note the symmetry of the 2 Sylow w.r.t. the eigenvalues.
Only case left is all odd. By parity invariance, suffices to check \( \vec{a} \equiv (1, 1, 1) \) to find \( d(M) \). \( d(M) = 2 \), so we need only determine largest 2-factor. WLOG \( a_1 + a_2 \equiv 2 \mod 4 \). Through algebraic manipulation we get that

\[
Syl_2(K(G)) \cong \mathbb{Z}/2^e \mathbb{Z} \oplus \mathbb{Z}/2^f \mathbb{Z}
\]

\[
e = v_2(a_2 + a_3) + 1, \quad f = v_2(a_1 + a_3) + 1
\]
Results for $r = 3$

**Proposition**
For $r = 3$, let $d_1 \leq d_2 \leq \cdots \leq d_7$ be all the powers of 2 in the nonzero eigenvalues of $L(G)$ for $M$ reduced. Let $c_{\text{top}}$ be the top Sylow-2 cyclic factor. Then

$$c_{\text{top}} = \begin{cases} 
2^{d_7+1} & \text{not all } d_i \text{ equal} \\
2^{d_7} & d_i = d_j \text{ for all } i, j \in \{1, \ldots, 7\}
\end{cases}.$$ 

**Theorem**
Let $G = G(\mathbb{F}_2^3, M)$ be generic, and with $d_i$ as above. Then

$$\text{Syl}_2(K(G)) = \begin{cases} 
\mathbb{Z}/2^{d_6-1}\mathbb{Z} \times (\mathbb{Z}/2^{d_7+1}\mathbb{Z})^2 & d_6 = d_7 \\
\mathbb{Z}/2^{d_5}\mathbb{Z} \times \mathbb{Z}/2^{d_6}\mathbb{Z} \times \mathbb{Z}/2^{d_7+1}\mathbb{Z} & d_6 < d_7
\end{cases}.$$
Investigating Largest Cyclic Factors of the Sandpile Group
In quotient ring of the hypercube sandpile group, Anzis and Prasad showed that $x_j - 1$ has maximal, finite, additive order for any $j \in \{1, \ldots, r\}$.

We adapted proof to show that for any generating set $(v_1, \ldots, v_m)$, the maximal order element of the form $x_{v_k} - 1$ has maximal finite order. By changing variables, we can assume that $x_1 - 1$ has maximal finite order.

From definition of cokernel, $\text{ord}(x_1 - 1)$ is smallest $C$ s.t.

$$\exists v \in \mathbb{Z}^2 \text{ s.t. } L(G)v = C(1, -1, 0, \ldots, 0) = Cw$$

here we use the isomorphism:

$$\mathbb{Z}^2 \cong \mathbb{Z}[x_1, \ldots, x_r]/(x_1^2 - 1, \ldots, x_r^2 - 1)$$
Top Cyclic Factor

**Theorem**

Let $d$ be the size of the largest cyclic factor in $K(G)$. Then $d | 2^{r-2} \lcm (\lambda_i : i \geq 2)$.

**Proof.**

An adaptation of the argument from Anzis and Prasad.

**Corollary**

The largest 2-cyclic factor, $\mathbb{Z}/2^e\mathbb{Z}$ has bound

$$e \leq \lceil \log_2(n) \rceil + r - 1$$

which is sharp when $G = Q_{2^k}, Q_{2^k+1}$.
Proof of Corollary.

Apply theorem while noting that the largest eigenvalue is bounded by $2n$, so that

$$v_2(d) \leq v_2 \left[ 2^{r-2} \text{lcm} (\lambda_i : i \geq 2) \right]$$

$$\leq r - 2 + \lfloor \log_2(2n) \rfloor = \lfloor \log_2(n) \rfloor + r - 1$$

when $G = Q_{2^k}$, we use the fact that each eigenvalue is distinct with largest value being $2^{k+1}$ and that $\lfloor \log_2(2^{k+1}) \rfloor = k + 1$. \qed
Main Result of Interest

We can actually improve the previous result:

**Corollary**

The order of $x_r - 1$ in $K(G)$ is equal to minimum integer $C$, such that for any $S \subseteq [n], |S| \geq 2, d \in \mathbb{F}_2^{|S|} \setminus \{0\}$,

$$C \frac{1}{2^{r-|S|}} \sum_{u_S=d} \frac{1}{\lambda_u} \in \mathbb{Z}$$
Specialization to $G = Q_n$

When $G = Q_n$ we know the eigenvalues and their multiplicities explicitly from Bai’s paper, so searching for $v \in \mathbb{F}_2^n$ and $C$ minimal such that $L(Q_n)v = Cw$ can be solved explicitly.

**Theorem**

For $n \geq 2$, let $c_n$ be the size of the largest cyclic factor in $K(Q_n)$. Then,

$$v_2(c_n) = \max\{\max_{x<n}\{v_2(x) + x\}, v_2(n) + n - 1\}.$$  

**Theorem**

For $n \geq 3$, the $2^{nd}$ to the $(n - 1)^{th}$ largest cyclic factor in $K(Q_n)$ all have the same size $d_n$. Moreover,

$$v_2(d_n) = \max_{x<n}\{v_2(x) + x\}.$$
Remaining Conjectures

Conjecture

For $n \geq 3$, let $e_n$ be the size of the $n^{th}$ largest cyclic factor in $K(Q_n)$. Then,

$$v_2(e_n) = \max \{ \max_{x < n-1} \{ v_2(x) + x \}, v_2(n-1) + n - 3 \}.$$ 

Similarly, for $n \geq 4$, let $f_n$ be the size of the $(n+1)^{th}$ largest cyclic factor in $K(Q_n)$. Then,

$$v_2(f_n) = \max_{x < n-1} \{ v_2(x) + x \}.$$
Section 5

Future Areas of Research
Groebner Bases

- Very difficult! Groebner bases must be redefined over \( \mathbb{Z} \), or in general PIDs vs. fields
- Recall we can order monomials \( x_I = \prod_{i \in I} x_i \) by the multi-indices they are indexed by
- For \( f = \sum_I a_I x_I = a_{I_0} x_{I_0} + \sum_{I \neq I_0} a_I x_I \) with \( x_{I_0} \) largest present, \( LT(f) = a_{I_0} x_{I_0}, \ lm(f) := x_{I_0}, \) and \( lc(f) = a_{I_0} \)
- Assuming a novel (unstated) definition of groebner basis, we have...
**Theorem**

For $A$ a PID, and ideal $s \subseteq A[x_1, \ldots, x_n]$. Let $G = \{g_i\}_{i=1}^t$ be a groebner basis for $s$. Let

$$J_{x\alpha} := \{i : \text{lm}(g_i) \mid x_{\alpha}, g_i \in G\}, \quad I_{J_{x\alpha}} := \langle\{\text{lc}(g_i) : i \in J_{x\alpha}\} \rangle$$

Call $I_{J_{x\alpha}}$ the leading coefficient ideal. Under a few other conditions (which hold for $A = \mathbb{Z}$), there exists an isomorphism

$$\phi : A[x_1, \ldots, x_n]/\langle G \rangle \rightarrow A/I_{J_{x\alpha,1}} \oplus \cdots \oplus A/I_{J_{x\alpha,m}}$$
Example

Consider

\[ M = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix} \]

\[ \{LT(g_i) \mid g_i \in G\} = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_3^2, 2x_1x_4, x_4^2, 6x_1, 24x_2, 24x_3, 480x_4\} \]

\[ K(G(F_2^n, M)) \cong (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/6\mathbb{Z}) \oplus (\mathbb{Z}/24\mathbb{Z})^4 \oplus (\mathbb{Z}/480\mathbb{Z}) \]

Note

\[ J_{x_2x_3} = \{9, 10\}, \quad I_{J_{x_2x_3}} = 24 \]

\[ J_{x_3x_4} = \{10, 11\}, \quad I_{J_{x_3x_4}} = 24 \]
Flaws with Groebner Basis Method

Sage's implemented version of groebner basis is not general enough for this isomorphism to always hold.

\[
M' = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

should yield same sandpile, but

\[
\{LT(g_i) | g_i \in G\} = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_3^2, x_1x_4, x_2x_4, 2x_3x_4, x_4^2, 24x_1, 24x_2, 48x_3, 60x_4\}
\]

which does not match the sandpile group (no order 480 term!). Sage is not to be trusted, but groebner bases could be useful in the future.
Matroid Contraction

For

\[
M = \begin{pmatrix}
| & \ldots & | \\
\vdots & \ddots & \vdots \\
| & \ldots & |
\end{pmatrix}
\]

where each \( v_i \in \mathbb{F}_2^r \), consider

\[
M' = \pi_{r-1}(M) = \begin{pmatrix}
| & \ldots & | \\
\pi_{r-1}(v_1) & \ddots & \pi_{r-1}(v_n) \\
| & \ldots & |
\end{pmatrix}
\]

\[
= \begin{pmatrix}
| & \ldots & | \\
v_1' & \ddots & v_n' \\
| & \ldots & |
\end{pmatrix}
\]
Continued

Gives rise to surjection

$$\mathbb{Z}[x_1, \ldots, x_r]/ \left( x_1^2 - 1, \ldots, x_r^2 - 1, n - \sum_{i=1}^{n} \prod_{j=1}^{r} x_j^{(v_i)_j} \right)$$

"$$x_r = 1$$" $\rightarrow$ $$\mathbb{Z}[x_1, \ldots, x_{r-1}]/ \left( x_1^2 - 1, \ldots, x_{r-1}^2 - 1, n' - \sum_{i=1}^{n} \prod_{j=1}^{r-1} x_j^{(v_i)_j} \right)$$

- Comparing torsion components: the cyclic factors in image can be viewed as subgroups of a larger cyclic factor in the domain sandpile group.
- Process of evaluating at $$x_r = 1$$ is matroid contraction.
Example

Consider

\[
M = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}
\mapsto M' = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

\[
[K(G(F^3_2, M)) = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/48\mathbb{Z} \oplus \mathbb{Z}/240\mathbb{Z}] \mapsto [K(G(F^2_2, M')) = \mathbb{Z}/24\mathbb{Z}]
\]

From Groebner basis approach, we can think of each invariant factor being generated by a monomial \( \bar{x}_i \). In fact...
Notice that \( \text{ord}(\bar{x}_I)_{M'} \mid \text{ord}(\bar{x}_I)_{M} \). Consistent with map of quotients

Indicates ”growth” of sandpile group
Future Work

Our data and preliminary results raise other questions:

- Can we find a bound from below the top cyclic factor of the sandpile group in terms of $r, n$? We have one for the cube, but not in general.
- Can we implement the novel definition of groebner bases for PIDs as described by Franz Pauer in his work "Groebner basis with coefficients in rings"?
- Can we show the sandpile group of a Cayley graph only depends on the set of eigenvalues, and not by their indexing set?
- Is there a larger pattern to the number of even invariant factors?
- Can we describe $r = 3$ in full generality? We have conjecture for all the cases except all odd parities
- Maybe $r = 4$ as well?

Unfortunately our funding has run out, so the world may never know...
Acknowledgements

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The End!
Questions?