Resistor Networks in a Punctured Disk

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Overview

1. Background Definitions and Results
   - Resistor Networks and Inverse Problem
   - Known Results: Circular Planar Resistor Networks

2. Resistor Networks on a Punctured Disk

3. Conjectures
A resistor network is a finite graph \((V, E)\) with a specified set \(B \subseteq V\) of boundary vertices and a real non-negative conductance \(c_e\), for each \(e \in E\). The remaining vertices, \(I = V - B\), are called internal vertices.
The Kirchoff Matrix $K(\Gamma)$ of a resistor network $\Gamma$ is the unique matrix with $K(\Gamma)_{ij}$ equal to the sum of conductances of edges between $i$ and $j$ and row sums equal to 0.
Example

$$K(\Gamma) = \begin{bmatrix}
-2 & 0 & 0 & 2 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & -4 & 3 & 0 & 1 \\
2 & 0 & 0 & 3 & -6 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 2 \\
0 & 1 & 0 & 1 & 0 & 2 & -4 \\
\end{bmatrix}$$
Response Matrix

**Definition**

A potential function assignment to the boundary vertices of $\Gamma$ induces a net current at boundary vertices. This may be represented by the *response matrix* of $\Gamma$, $\Lambda(\Gamma)$. 
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The Response Matrix can be calculated in terms of the Kirchoff matrix:

$$\Lambda = A - BC^{-1}B^t$$
Example

\[ K(\Gamma) = \begin{bmatrix} -2 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -4 & 3 & 0 & 1 \\ 2 & 0 & 0 & 3 & -6 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 2 \\ 0 & 1 & 0 & 1 & 0 & 2 & -4 \end{bmatrix} \]

\[ \Lambda(\Gamma) = \begin{bmatrix} -22 & 1 & 2 & 19 \\ \frac{1}{17} & \frac{1}{17} & \frac{1}{17} & \frac{1}{17} \\ 1 & -7 & -1 & 3 \\ \frac{1}{17} & \frac{1}{17} & \frac{1}{17} + \frac{1}{4} \\ \frac{2}{17} & \frac{3}{17} & -\frac{11}{17} & 6 \\ \frac{19}{17} & \frac{3}{17} + \frac{1}{4} & 6 & \frac{28}{17} + \frac{1}{4} \\ \frac{17}{17} & \frac{17}{4} & \frac{17}{17} & -\frac{17}{4} \end{bmatrix} \]
Inverse Problem

Given a resistor network $\Gamma$ without labeled conductances and $\Lambda(\Gamma)$, when are we able to uniquely recover its conductances?
Curtis, Ingerman, and Morrow solved the inverse problem for a special class of graphs known as circular planar resistor networks (cprns).

**Definition**

A *circular planar resistor network* is a resistor network that can be embedded in a disk so that it is planar with all boundary vertices are on the boundary of the disk.
Circular Planar Resistor Networks

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**Example**

![Figure: A cprn](image)
Constructing the Medial Graph
Constructing the Medial Graph

Figure: Add in new vertices
Constructing the Medial Graph

Figure: Connect Edges
Constructing the Medial Graph

Figure: 4 Strands of the Medial Graph
Z-Sequence

Figure: The z-sequence of this network is 1 2 3 1 4 2 3 4
Definition

Call two resistor networks $\Gamma$ and $\Gamma'$ *electrically equivalent* if the following holds:

- For every assignment of conductances to $\Gamma$, there exists an assignment of conductances to $\Gamma'$ such that $\Lambda(\Gamma) = \Lambda(\Gamma')$.
- For every assignment of conductances to $\Gamma'$, there exists an assignment of conductances to $\Gamma$ such that $\Lambda(\Gamma) = \Lambda(\Gamma')$.
Local Transformations

The following transformations can be done without affecting the response matrix:

1. Parallel Reduction

2. Series Reduction

3. Pendant Removal
Local Transformations (continued)

- $\gamma - \Delta$ moves

Diagram showing the transformation of a graph from one configuration to another, illustrating the $\gamma - \Delta$ move.
Definition

Call a cprn *critical* if it is not electrically equivalent to any graph with fewer edges.
Theorem (Curtis, Ingerman, Morrow)

A cprn is critical if and only if it satisfies the following medial graph conditions:

- No medial strands form closed loops.
- No medial strands self-intersect.
- No two medial strands intersect more than once.

Furthermore, For two critical circular planar resistance networks $\Gamma_1$ and $\Gamma_2$, the following conditions are equivalent:

- $\Gamma_1$ and $\Gamma_2$ are electrically equivalent.
- $\Gamma_1$ and $\Gamma_2$ are related by $Y - \Delta$ moves.
- $\Gamma_1$ and $\Gamma_2$ share a z-sequence.
Theorem

We can uniquely recover the conductances of a CPRN if and only if it is critical. Additionally, every CPRN can be transformed to a critical CPRN through a sequence of the defined local moves.
Resistor Networks on a Punctured Disk

We worked towards expanding on Curtis, Ingerman, and Morrow’s results by examining a new class of resistor networks.

**Definition**

A *Resistor Network on a Punctured Disk* (rnpd) is a resistor network that can be embedded in a disk so that it is planar, and all boundary vertices but one (the *interior boundary vertex*) are on the boundary of the disk.

**Example**
The Medial Graph and $Z$-sequences for RNPDs

**Definition**

The *medial graph for an rnpd* is the medial graph of the cprn that results from treating the interior boundary vertex as internal.

**Definition**

The *$z$-sequence for an rnpd* is defined similarly as for cprns, with a slight modification. In the medial graph, we label one endpoint of each strand $s$ with an $s'$, such from the perspective of the interior boundary vertex the strand travels clockwise from $s$ to $s'$. Additionally, if a strand $s$ contains a self-intersection, underline $s'$. 
Z-sequence Illustration
Figure: Z-Sequence: 1' 2 3' 1 2' 4 5' 3 3 4' 5
Irreducible RNPD results

**Definition**

We call an rnpd *irreducible* if it is not electrically equivalent to an rnpd with fewer edges.

**Theorem**

In any irreducible rnpd,

- No medial strand is a closed circle.
- Every medial lens and medial loop contains the interior boundary vertex.
- Every strand intersects itself at most once.
- At most one strand contains a self-intersection.
- Every pair of strands intersects at most twice.
Irreducible RNPD results

Theorem

Two irreducible rnps share a z-sequence if and only if they are related by $Y - \Delta$ moves.
4-Periodic Graphs

We define an infinite family of cprns called 4-periodic graphs

\[ \Pi_3 \]

\[ \Pi_4 \]

\[ \Pi_5 \]

\[ \Pi_6 \]
Properties of 4-periodic graphs:

- Critical cprns with $z$-sequence $1 \ 2 \ \cdots \ \ n \ 1 \ 2 \ \cdots \ \ n$
  (Electrically equivalent to special network in cprn case: $\Sigma_n$)
- Maximal critical cprns
Spider Graphs

We construct a new family of graphs known as *spider graphs* from 4-periodic graphs.
Spider Graphs

Theorem

*Spider Graphs are recoverable*

**Terms:**
- **Boundary Edge**: An edge connecting two boundary vertices
- **Boundary Spike**: An edge connecting an internal vertex to a boundary vertex of degree 1

We say $P, Q \subseteq B$ form a connection $(P, Q)$ if $|P| = |Q| = k$ and there exist $k$ disjoint paths through internal vertices connecting each $p \in P$ to a $q \in Q$.
Spider Graphs

Theorem

*Spider Graphs are recoverable*

Proof Idea

Terms:

- **Boundary Edge**: An edge connecting two boundary vertices
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We say $P, Q \subseteq B$ form a connection $(P, Q)$ if $|P| = |Q| = k$ and there exist $k$ disjoint paths through internal vertices connecting each $p \in P$ to a $q \in Q$
Theorem

Spider Graphs are recoverable

Proof Idea

Known for cprns: If deleting or contracting a boundary edge or spike breaks some connection, we can recover the conductance of that edge or spike from the response matrix.

We generalized this result for rnpds by restricting $P$ and $Q$ to not contain the interior boundary vertex.
Theorem

Spider Graphs are recoverable

Proof Idea

Deleting any boundary edge or boundary spike in our spider graph results in a broken connection (because 4-periodic graphs are critical).

We can delete and contract boundary edges and spikes one by one, until we are left with a *star graph*, which is trivially recoverable.

![Star Graph]

*Figure: Star Graph*
Sufficient Condition for Recoverability

We can use the same process to generalize our result for Spider Graphs to a much larger family of rnpds:

**Theorem**

Let $\Gamma$ be any critical cprn. Let $\Gamma'$ be the result of inserting a star graph into one of the faces of $\Gamma$. Then, $\Gamma'$ is a recoverable rnpd.
Necessary Condition for Recoverability of RNPDs

Algorithm:
- For rnpd $\Gamma$: Remove interior boundary vertex and change all its neighbors to boundary vertices. Repeat process for each newly created interior boundary vertex until you get a cprn. If the original rnpd was recoverable, then the resulting cprn is.
Additional Local Moves for RNPDs

The following moves can be done in a way that does not affect the response matrix:

Figure: Antenna Absorption

Figure: Antenna Jumping
Additional Local Moves for RNPDs

The following are local move equivalences that alter z-sequences.

Figure: Square Jump

Figure: Generalized Antenna Absorption
Conjectures

- An rnpd is recoverable if and only if it is irreducible (in which case we'd have a natural definition of *critical*).
- The moves described in the talk are sufficient to describe all electrical equivalences of rnpds.
References

Circular Planar Graphs and Resistor Networks
Linear Algebra and its Applications 283, pgs 115 - 150
Questions?