Algebraic Monoids and Their Hecke Algebras

Jared Marx-Kuo, Vaughan McDonald, John M. O’Brien, & Alexander Vetter

University of Minnesota REU in Algebraic Combinatorics

August 2, 2018
Outline

1 Introduction
   - Background on Monoids
   - Examples of Renner Monoids

2 Monoid Representation Theory
   - Definitions
   - Induced Representations

3 Representations of Renner Monoids
   - Rook Monoid Representations
   - Symplectic Rook Monoid Representations

4 Hecke algebras of monoids
   - The Borel-Matsumoto theorem for finite monoids

5 References
In this presentation, we explore algebraic monoids, their Hecke algebras, and their representations.

We seek to produce analogous results from finite algebraic group representation theory in the setting of algebraic monoids.

We focus on the representation theory of the rook monoid $R_n$ and the symplectic rook monoid $RSp_{2n}$, and their Hecke algebras, $\mathcal{H}(R_n)$ and $\mathcal{H}(RSp_{2n})$, respectively.
Background on Monoids

Definition

A monoid is a semigroup (assoc. mult.) with identity.

Contained in every monoid, $M$, is a group of units (i.e., invertible elements) $G(M)$. By studying $M$, we gain valuable insight into the action of $G(M)$, informing its representation theory.
Background on Monoids

**Definition**

A **monoid** is a semigroup (assoc. mult.) with identity.

Contained in every monoid, $M$, is a group of units (i.e., invertible elements) $G(M)$. By studying $M$, we gain valuable insight into the action of $G(M)$, informing its representation theory.

**Definition**

$M$ is an algebraic monoid if it is a Zariski-closed subset of $\text{Mat}_n(F)$ for some $n \in \mathbb{Z}$ and $F$ a field. Furthermore, $M$ is **reductive** if $G(M)$ is a reductive group and $M$ is an irreducible algebraic variety.
Properties of reductive monoids

If $M$ is reductive, $G(M)$ has a Borel subgroup $B$, e.g. the invertible upper triangular matrices in the case of $\text{Mat}_n(F)$.

Furthermore, $M$ has a Renner decomposition as the disjoint union of double cosets of $B$:

$$M = \bigsqcup_{r \in R} BrB$$  \hspace{1cm} (1)

where $R$, the Renner monoid of $M$, encodes vital structural information about $M$.

The group of units of $R$ is the Weyl group of $G(M)$. Furthermore, $R$ has the decomposition

$$R = G(R)E(\overline{T})$$  \hspace{1cm} (2)

where $E(\overline{T})$ is a set of idempotents.
Rook Monoid

The “Rook Monoid” is the Renner monoid of the algebraic monoid $\text{Mat}_n(F)$.

- $R_n$ is realized as the set of all $n \times n$ matrices with entries 0 and 1 such that each row and column has at most one nonzero entry.
- We call this the Rook monoid because if we view the ones as rooks, then this monoid is the set of all $n \times n$ chessboard with at most $n$ non-attacking rooks.
- Its unit group $G(R_n)$ is isomorphic to the symmetric group, $S_n$. 
### Rook Monoid Examples

**Example**

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0 \\
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{pmatrix} \in R_3
\]
Rook Monoid Examples

Example

\[
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix} \in R_3
\]

Example (er... Non-example)

\[
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
0 & 1 & 0
\end{pmatrix}, \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}, \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 1 \\
0 & 0 & 1
\end{pmatrix} \notin R_3
\]
Similarly, the symplectic Rook monoid is the Renner monoid for the more complicated algebraic monoid whose unit group is the symplectic group $\text{Sp}_{2n}(F)$. Further, The $B_n$ Weyl group embeds as $G(\text{RSp}_{2n})$.

Nice presentation:

**Theorem**

$$\text{RSp}_{2n} \cong \{ A \in R_{2n} \mid AJA^T = 0 \text{ or } J \}, \quad J = \begin{pmatrix} 0 & \ldots & 0 & 1 \\ 0 & \ldots & 1 & 0 \\ \vdots & \ddots & \ddots \\ 1 & 0 & \ldots & 0 \end{pmatrix}$$
Symplectic Rook Monoid Examples

Example

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in RSp_4
Symplectic Rook Monoid Examples

**Example**

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \in \mathcal{RS}_4
\]

---

**Example (er... Non-example)**

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \notin \mathcal{RS}_4
\]
Representations of Monoids

Let $M, N$ be monoids. A map $\varphi : M \to N$ is a homomorphism of monoids if the following hold:

- For all $m_i \in M$, $\pi(m_1 m_2) = \pi(m_1) \pi(m_2)$.
- For $e_M, e_N$ the identity elements of $M$ and $N$ respectively, $\pi(e_M) = e_N$.

Let $V$ be a vector space over $k$. A morphism $\pi : M \to \text{End}_k(V)$ is called a representation of $M$. We denote representations as the pair $(\pi, V)$.

A representation is irreducible if it has no proper subrepresentations.

If $V$ is finite dimensional, we define the character $\chi : M \to k$ of $\pi$ as the function defined by $\chi(m) = \text{tr}(\pi(m))$ for all $m \in M$. 


Induced Representations

Let $N$ be a submonoid of $M$ and $(\pi, V)$ a representation of $N$. We have that $(\pi, V)$ induces a representation $(\text{Ind}_{N}^{M} \pi, \text{Ind}_{N}^{M} V)$ of $M$. Define

- $\text{Ind}_{N}^{M} V = \{ f : M \to V \mid f(nm) = \pi(n)f(m) \} \quad \forall n \in N, m \in M$
- $(\text{Ind}_{N}^{M} \pi)(m)f(x) = f(xm) \quad \forall x, m \in M$.

We proved that the following result holds in the case of monoids:

Frobenius Reciprocity for finite monoids

If $N$ is a submonoid of $M$, $(\pi, V)$ a representation of $N$, and $(\sigma, W)$ a representation of $M$, then

$$\text{Hom}_{M}(\text{Ind}_{N}^{M} V, W) \cong \text{Hom}_{N}(V, W) \quad (3)$$

as vector spaces over $F$.
The irreducible representations of $R_n$ are indexed by partitions of at most $n$.

Further, these representations are derived from representations of $S_k$ for $k \in \{0, \ldots, n\}$.

Let $\lambda$ be a partition of $k$, and let $V^\lambda$ be the corresponding irreducible representation of $S_k$.

- There exists an irreducible representation $W^\lambda$ of $R_n$.
- $\dim(W^\lambda) = \binom{n}{k} \dim(V^\lambda)$
The irreducible representations of $R_n$ are indexed by partitions of at most $n$.

Further, these representations are derived from representations of $S_k$ for $k \in \{0, \ldots, n\}$.

Let $\lambda$ be a partition of $k$, and let $V^\lambda$ be the corresponding irreducible representation of $S_k$.

- There exists an irreducible representation $W^\lambda$ of $R_n$.
  - $\dim(W^\lambda) = \binom{n}{k} \dim(V^\lambda)$

We note that “conjugacy classes” of the monoid are also indexed by partitions of at most $n$.

It turns out the character table of any Renner monoid is block upper triangular, when the representations are the columns and conjugacy classes are the rows.
Let $Ch_k$ be the character table of $S_k$. Then define $Y_n$ to be the following block diagonal matrix:

\[
Y_n = \begin{pmatrix}
Ch_n & Ch_{n-1} & \cdots & Ch_1 & Ch_0 \\
Ch_{n-1} & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
Ch_1 & \ddots & \ddots & \ddots & \ddots \\
Ch_0 & \ddots & \ddots & \ddots & \ddots
\end{pmatrix}
\]

Let $M_n$ be the character table of $R_n$. Solomon found explicit descriptions of the matrices $A$ and $B$ such that

\[
M_n = AY_n = Y_nB
\]  

(4)

The $A$ matrix comes from combinatorics of cycle structures.

The $B$ matrix comes from the Pieri rules for induced representations.
Our motivation in this section comes from restricting our monoid representations to their corresponding group of units. Using [Solomon, 2002] and [Li et al., 2008], we obtain the following result:

**Theorem**

Let $W_n$ be a Weyl group of type $A_n, B_n, C_n, \text{ or } D_n,$ with corresponding Renner monoids $RW_n$. Let $\chi$ be a character of $S_r$, and $\chi^*$ the associated character of $W_n$. Then

$$\chi^*|_{W_n} = \text{Ind}_{S_k \times W_{n-k}}^{W_n} (\chi \otimes \eta_{n-k})$$

In particular, when the Weyl group is $A_n$, the above restriction produces the well-known Pieri rules. From this result, we can now describe the $B$ matrix as Solomon does.
Let $\lambda$ and $\mu$ index partitions of at most $n$. Recall that the rows and columns were also indexed by partitions. Thus, we can describe the $B$ matrix entries by the partitions. Solomon finds the $B$ matrix to be:

$$B_{\lambda,\mu} = \begin{cases} 1, & \text{if } \lambda - \mu \text{ is a horizontal strip} \\ 0, & \text{otherwise} \end{cases}$$

This comes exactly from the Pieri rules for type A found in [Geck et al., 2000].
Example from $R_3$ Character Table

\[ M_3 = \begin{pmatrix} 1 & 2 & 1 & 3 & 3 & 3 & 1 \\ 1 & 0 & -1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

\[ Y_3B_3 = \begin{pmatrix} 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]
Symplectic Rook Monoid Representations

- Similar story in the Symplectic Rook monoid case.
- The irreducible representations of $RSp_{2n}$ are indexed by pairs of partitions, $(\lambda, \mu)$, such that $|\lambda| + |\mu| = n$, as well as partitions, $\nu$, of $\{0, \ldots, n\}$.
- The representations are derived from representations of $B_n$ and $S_k$.
Symplectic Rook Monoid Representations

- Similar story in the Symplectic Rook monoid case.

- The irreducible representations of $RSp_{2n}$ are indexed by pairs of partitions, $(\lambda, \mu)$, such that $|\lambda| + |\mu| = n$, as well as partitions, $\nu$, of $\{0, \ldots, n\}$.

- The representations are derived from representations of $B_n$ and $S_k$.

- Let $(\lambda, \mu)$ be as above, and let $V^{(\lambda, \mu)}$ be the corresponding irreducible representation of $B_n$.
  - There exists an irreducible representation $W^{(\lambda, \mu)}$ of $RSp_{2n}$.

- Let $\nu$ be as above, and let $V^\nu$ be the corresponding irreducible representation of $S_k$.
  - There exists an irreducible representation $W^\nu$ of $RSp_{2n}$.
  - $\dim(W^\nu) = 2^k \binom{n}{k} \dim(V^\nu)$

- We note that “conjugacy classes” of the monoid are also indexed by partitions of at most $n$ and pairs of partitions whose sum is $n$. 
Character Table of $RSp_{2n}$

Let $X_n$ be the character table of $B_n$, and let $Ch_k$ be the character table of $S_k$. Then define $Y_n$ to be the following block diagonal matrix:

$$
Y_n = \begin{pmatrix}
X_n & & & & \\
& Ch_n & & & \\
& & Ch_{n-1} & & \\
& & & \cdots & \\
& & & & Ch_1 \\
& & & & & Ch_0
\end{pmatrix}
$$

Let $CRSp_{2n}$ be the character table of $RSp_{2n}$. In the spirit of Solomon, we derive explicit descriptions of the matrices A and B such that

$$
CRSp_{2n} = AY_n = Y_n B \quad (5)
$$

The $A$ matrix comes from combinatorics of cycle structures.

The $B$ matrix comes from the Pieri rules for induced representations.
B matrix for $RSp_{2n}$

We determine the character table to be the following:

$$CRSp_{2n} = \begin{bmatrix} X_n & \ast \\ 0 & M_n \end{bmatrix}$$  \hspace{1cm} (6)

We are able to determine the B matrix in a similar way to the rook matrix. In particular:

$$B = \begin{bmatrix} \text{Id} & P \\ 0 & B^* \end{bmatrix}$$  \hspace{1cm} (7)

where $B^*$ is the B matrix for $R_n$, and P comes from Pieri rules in the type B case.
Pieri Coefficients for type B

**Theorem**

Let \( \nu \vdash k \) index a representation of \( S_k \). Then,

\[
\text{Ind}^B_{S_k \times B_{n-k}}(\tau_{\nu} \boxtimes \eta_{n-k}) = \sum_{\gamma, \mu} \left( \sum_{\gamma + \mu \vdash n, \lambda \vdash n-k \text{ horiz. strip}} c^\nu_{\lambda, \mu} \right) \chi_{\gamma, \mu} \tag{8}
\]

The coefficients obtained from the above formula are the numbers in the P matrix on the previous slide.
What the Hecke?

- It turns out, we can form Hecke algebras from $R_n$ and $RSp_{2n}$.
- $\mathcal{H}(R_n)$
  - Representations of $\mathcal{H}(R_n)$ are described by [Halverson, 2004].
  - The character table is described in [Dieng et al., 2003].
  - We show that the character table can be decomposed into
    
    $$\mathcal{M}_n = Y_n B$$  \hspace{1cm} (9)

    where $Y_n$ is a block diagonal matrix with Hecke algebra character table blocks, and $B$ is the same $B$ matrix we computed for $R_n$. 

It turns out, we can form Hecke algebras from $R_n$ and $RSp_{2n}$.

\[ H(R_n) \]
- Representations of $H(R_n)$ are described by [Halverson, 2004].
- The character table is described in [Dieng et al., 2003].
- We show that the character table can be decomposed into

\[ M_n = Y_n B \] (9)

where $Y_n$ is a block diagonal matrix with Hecke algebra character table blocks, and $B$ is the same $B$ matrix we computed for $R_n$.

\[ H(RSp_{2n}) \]
- Representations have not been described before.
- We give a first description of the character table.
- We show that the character table can be decomposed into

\[ M_{2n} = Y_n B \] (10)

where $Y_n$ is a block diagonal matrix with Hecke algebra character table blocks, and $B$ is the same $B$ matrix we computed for $RSp_{2n}$.
The Iwahori-Hecke algebra of a reductive monoid

Let $M$ be a reductive monoid over a finite field $F$. Recall that $M$ unit group $G(M)$, Borel subgroup $B$, and Renner monoid $R$.

**Definition**

The **Hecke algebra** $\mathcal{H}(M, B)$ over $\mathbb{C}$ is the algebra

$$\mathcal{H}(M, B) = \{ f : M \to \mathbb{C} \mid f(b_1 xb_2) = f(x) \ \forall b_1, b_2 \in B, \ x \in M \} \quad (11)$$

under addition and convolution of functions, with convolution given by

$$(f * g)(x) = \sum_{yz=x} f(y)g(z). \quad (12)$$
Properties of Hecke algebras

- The Hecke algebra of a monoid has a basis over $\mathbb{C}$ given by, for all $r \in R$, $1_{BrB}$ defined to be the characteristic function of the double coset of $r$.

- Let $M$ be a reductive monoid with Renner monoid $R$. Then $\mathcal{H}(M, B) \cong \mathbb{C}[R]$ as $\mathbb{C}$-algebras.

- Let $(\pi, V)$ be a representation of $M$. Then $V$ has a $\mathcal{H}(M, B)$-module structure under the following action: for $f \in \mathcal{H}(M, B)$,

  $$\pi(f)v = \sum_{x \in M} f(x)\pi(x)v$$  \hspace{1cm} (13)

- Let $V^B = \{v \in V \mid \pi(b)v = v \ \forall b \in B\}$ be the space of vectors fixed pointwise by a Borel subgroup. The Hecke algebra of an algebraic monoid $M$ encodes information about representations of $M$ with $V^B$ nonzero.
The Borel-Matsumoto Theorem

The Borel-Matsumoto theorem for finite monoids

- Let $(\pi, V)$ be an irreducible representation of $M$ with $V^B \neq \{0\}$. Then $V^B$ is irreducible as an $\mathcal{H}(M, B)$-module.
- If $(\pi, V)$ and $(\sigma, W)$ are two irreducible representations of $M$ with $V^B$ and $W^B$ nonzero and isomorphic as $\mathcal{H}(M, B)$-modules, then $(\pi, V) \cong (\sigma, W)$.

The Borel-Matsumoto theorem allows us to reduce questions about representations of our algebraic monoid $M$ with $V^B$ nonzero to questions about the representations of $\mathcal{H}(M, B)$.

Since $\mathcal{H}(M, B) \cong \mathbb{C}[R]$ for $R$, the Renner monoid of $M$, its representation theory is markedly simpler than that of $M$ itself.

In theory, we could use $\mathcal{H}(M, B)$ to classify irreducible representations of $M$ with $V^B$ nonzero.
Further Questions

- How do representations of $R_{2n}$ restrict to $RSp_{2n}$?
- What does this process look like for type $D$ Renner monoids?
- Can we construct the irreducible representations of a reductive monoid $M$ with $V^B$ nonzero guaranteed by the Borel-Matsumoto theorem?
- Is there a Deligne-Lusztig theory for finite monoids of Lie type?
- Is there a Borel-Matsumoto theorem for $p$-adic reductive monoids?
- Does the comparatively simple geometry of algebraic monoids help us with their representation theory?
- What other aspects of the theory of group Hecke algebras hold in the case of monoid Hecke algebras?
References I

- **Bump, D. (2011).**
  Hecke algebras.

- **Dieng, M., Halverson, T., and Poladian, V. (2003).**
  Character formulas for q-rook monoid algebras.

- **Geck, M., Pfeiffer, G., et al. (2000).**
  *Characters of finite Coxeter groups and Iwahori-Hecke algebras.*

- **Godelle, E. (2010).**
  Generic hecke algebra for renner monoids.
*Graphs, dioids and semirings: new models and algorithms*, volume 41.  

Representations of the q-rook monoid.  

Representations of the symplectic rook monoid.  

An introduction to reductive monoids.  
*NATO ASI Series C Mathematical and Physical Sciences-Advanced Study Institute*, 466:295–352.
Acknowledgements

Special thanks to our mentor Dr. Ben Brubaker and TA Andy Hardt for guiding us on this project.
Questions

Any questions?