Let $G = (V, E)$ be a directed edge-weighted graph with no loops or multiple edges.

Consider a variable $x_v$ for each vertex $v \in V$. 

**R-systems**
Given \((x_v)_{v \in V}\), we want to find a new set of variables \((x'_v)_{v \in V}\) that satisfy

\[
(x_v x'_v)^{(\ast)} = \left( \sum_{(u,v) \in E} \frac{w_t(u,v)}{x'_v} \right)^{-1} \left( \sum_{(v,w) \in E} w_t(v,w) x'_w \right)
\]

More symmetrically,

\[
\sum_{(u,v) \in E} w_t(u,v) \frac{x'_v}{x'_u} = \sum_{(v,w) \in E} w_t(v,w) \frac{x'_w}{x'_v}
\]
Example

(edges with no weight are \( \uparrow \))

Check bottom left vertex:

\[
\text{LHS}(\ast) = abc
\]

\[
\text{RHS}(\ast) = b \left( \frac{1}{ac} \right)^{-1} = abc
\]

Check top left vertex:

\[
\text{RHS}(\ast) = bc(c+d)
\]

\[
\text{LHS}(\ast) = (c+d)\left( \frac{1}{bc} \right)^{-1}
\]
Why do this?

This generalizes a process called "birational rowmotion" defined by Einstein & Propp,

david, not albert

- an operation on (variables assigned to elements of) posets
- combines ideas of birational toggling (a well-studied operation) and rowmotion (toggling elements in and out of poset ideals)

Consider a poset (= partially ordered set) as a directed graph with all edges oriented upward.
Add an additional vertex \( s \).

Add edges
- from \( s \) to any source
- from any sink to \( s \)

\[ \text{e.g.} \]

Applying the transformation
\( (x_i) \rightarrow (x'_i) \) where we fix \( x_s = x'_s = 1 \)

is the same as doing birational rowmotion on the poset.
**DEF'N:** A graph $G = (V, E)$ is strongly connected if $\forall u, v \in V \exists$ a directed path $u \to v$ in $G$.

**DEF'N:** For a strongly connected $G$, an arborescence rooted at $v$ is a spanning tree directed toward $v$.

**EXAMPLE:** has one rooted at 1:

```
3 \rightarrow 1
```

but two rooted at 3:

```
3 \rightarrow 1
```

```
3 \rightarrow 1
```
**DEFN:** The weight of an arborescence $T$ is

$$w(T, X) := \prod_{(u,v) \in T} \frac{x_v}{x_u}$$

$$T = \begin{array}{c}
\sqrt{2} \\
\frac{3}{\rightarrow 1}
\end{array}$$

$$w(T, X) = \left(\frac{x_1}{x_2}\right) \left(\frac{x_2}{x_3}\right)$$
Theorem (Galashin-Pylyavskyy, 2017)

Let $G = (V, E)$ be strongly connected. Given $(x_r)_{r \in V}$, there exists a unique solution for $(x'_r)_{r \in V}$ up to rescaling:

$$x'_v = \frac{x_v}{\sum_{\text{arborescences } T \text{ rooted at } v} \text{wt}(T, x)}$$

**Example:** $G$ hom before has

$$x'_1 = \frac{x_1}{\left(\frac{x_1}{x_2}\right)} = x_2$$

$$x'_2 = \frac{x_2}{\left(\frac{x_2}{x_3}\right)} = x_3$$

$$x'_3 = \frac{x_3}{\frac{x_3}{x_1} + \frac{x_3^2}{x_2 + x_3}} = \frac{x_1 x_2}{x_2 + x_3}$$
Exercise 6

Consider the graph

\begin{center}
\begin{tikzpicture}
\node[vertex] (a) at (-2.5,0) {a};
\node[vertex] (b) at (2.5,0) {b};
\node[vertex] (c) at (0,2.5) {c};
\node[vertex] (d) at (2.5,5) {d};
\node[vertex] (e) at (0,5) {e};
\path[->,thick]
(a) edge (e);
(e) edge (b);
(e) edge (c);
(c) edge (d);
(d) edge (b);
\end{tikzpicture}
\end{center}

use the arborescence formula to find \((x_v')\) corresponding to \((x_v) = (a,b,c,d,e)\) above.
**DEFINITION:** For a strongly connected $G$, define $\phi$ to be the map $(x_v) \rightarrow (x'_v)$. Then the **R-system** associated with $G$ is the discrete dynamical system obtained by iterating $\phi$.

**DEFINITION:** A **rectangle poset** is the product of two chain posets.
THM (Grinberg-Roby) 2016

Birational rowmotion is periodic on rectangular posets, with period $ptg$.

This can be proven using a "T sequence"
T-sequence idea: write our $X_i$ variables as Laurent monomials (= monomials with positive and negative exponents allowed) in some other variables $\{T_i\}_{i \in \mathbb{Z}}$, where the $T_i$'s form a recursive sequence and each $T_i$ is a Laurent polynomial in the original values.
EXAMPLE:

\[
\begin{array}{c}
\left( x_1, x_2, x_3 \right) \\
U_1 \quad U_2 \quad U_3
\end{array}
\rightarrow
\begin{array}{c}
\left( x_2, x_3, \frac{x_1 x_2}{x_2 + x_3} \right) \\
U_2 \quad U_3 \quad U_4
\end{array}
\rightarrow
\begin{array}{c}
U_3 \quad U_4 \quad U_5
\end{array}
\rightarrow \ldots
\]

Let \( T_0 = x_1 x_2 x_3 \)

\( T_1 = x_2 x_3 \)

\( T_2 = x_3 \)

\( T_3 = 1 \)

and \( T_n T_{n+4} = T_{n+1} T_{n+3} + T_{n+2}^2 \)

the \textbf{Somos-4 sequence}, known to have Laurens phenomenon (can write all \( T_i \)'s as Laurent polynomials in \( T_0, T_1, T_2, T_3 \))
CLAIM: \[ u_n = \frac{T_n}{T_{n+1}} \]

\[ U_{n+3} = \frac{T_{n+3}}{T_{n+4}} = \frac{T_n}{T_{n+1}T_{n+3} + T_{n+2}} \]

versus

\[ U_{n+3} = \frac{u_n u_{n+1}}{u_{n+1} + u_{n+2}} = \frac{\left(\frac{T_n}{T_{n+1}}\right)\left(\frac{T_{n+1}}{T_{n+2}}\right)}{\left(\frac{T_{n+1}}{T_{n+2}}\right) + \left(\frac{T_{n+2}}{T_{n+3}}\right)} \]

\[ = \ldots = \text{expression above} \]

\( \checkmark \)

from the map \( \phi \)

induction
Back to rectangles

There is a \( \tau \)-sequence

\[
\alpha_{ij}(t) \alpha_{ij}(t+1) = \begin{cases} 
\text{see Sunita's scanned notes!} 
\end{cases}
\]

and \( x_{ij}(t) = \frac{\alpha_{i+1,j+1}(t)}{\alpha_{ij}(t+1)} \)
REU Exercise 7:
(a) List the relations that must be true for this system to be consistent.
(b) Check the relations for
   \[2 \leq i \leq p\]
   \[2 \leq j \leq q\]
We can prove periodicity using this \( T \)-sequence (part of this will be done in this afternoon's TA session).

**DEF'N**: A **trapezoid poset** is a rectangle poset with the sides cut off by diagonals.
REU Problem 3a

Prove birational rowmotion is periodic on all trapezoid posets (ask Sylvester what the conjectural period is).
**DEFN:** A \( k \)-fold cover of a graph \( G=(V_G,E_G) \) is a graph \( H=(V_H,E_H) \) with a map \( \pi: H \rightarrow G \) such that

1. \( \pi \) maps vertices to vertices, edges to edges.
2. \( |\pi^{-1}(v)| = |\pi^{-1}(e)| = k \quad \forall v \in V_G, e \in E_G \)
3. For \( e=(u,v) \in E_H \),
   \[ \pi(e) = (\pi(u), \pi(v)) \in E_G \]
4. For \( e \in E_H \), \( wt(e) = wt(\pi(e)) \).
EXAMPLE:

\[1' \xrightarrow{a} 2' \]
\[1'' \xrightarrow{a} 2'' \]
\[1'' \xrightarrow{a} 2'' \]

\[H \quad \pi \quad G\]
**Thm:** If $H$ is a $k$-fold cover of $G$ and $v'$ covers $v$, then

\[
\sum_{\text{arborescences } T \text{ rooted at } v \text{ in } G} \sum' \text{wt}(T) \quad \text{does not depend upon the choice of } v', \text{ nor on } v.
\]

In above example, picking $v=1, v'=1'$

get \( \frac{c+d}{c+da+bc+db+ad} = \frac{1}{bc+ad} \)

but picking $v=2, v'=2'$

get \( \frac{a+b}{ada+cba+dab+bcb} = \frac{1}{bc+ad} \)
REU Problem 3b

Write this ratio as a determinant, and investigate the positivity of the coefficients.

Why might one think it is related to a determinant?

**DEFN:** The Laplacian matrix of a graph $G = (V, E)$ is $L = (L_{ij})_{1 \leq i, j \leq n}$, where

$$L_{ij} = \begin{cases} \sum_{e \text{ with source } v_i} \text{wt}(e) & \text{if } i = j \\ - \sum_{e = (v_i, v_j)} \text{wt}(e) & \text{if } i \neq j \end{cases}$$
EXAMPLE: \( G \) from before

\[
L = \begin{bmatrix}
    a+b & -a-b \\
    -c-d & c+d
\end{bmatrix}
\]

Consider the 2-fold cover case, and label one lift of each vertex as \( \Theta \), the other as \( \Theta' \). Call an edge of \( G \) positive if its lifts are \( \Theta \rightarrow \Theta' \), negative if its lifts are \( \Theta \rightarrow \Theta \).
In the Laplacian of $G$, switch the sign of the weights of the positive edges off the diagonal.

**EXAMPLE**

$$
\begin{bmatrix}
a+b & a-b \\
c-d & c+d
\end{bmatrix}
$$

**"CONJECTURE"**

Our ratio is $\left[ \frac{1}{2} \text{det of this} \right]^{-1}$
EXAMPLE

\[
\begin{vmatrix}
 a+b & a-b \\
 c-d & c+d
\end{vmatrix} = (a+b)(c+d) - (a-b)(c-d)
\]

\[
= ac + ad + bc + bd - ac + ad + bc - bd
\]

\[
= 2(ad + bc)
\]

REM Exercise 8
Check the conjecture for

\[ G = \]

\[ H = \]

(maybe it fails?)