Ribbon Lattices and Ribbon Functions

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Boundary Conditions: For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ create a grid with $r$ rows and $\lambda_1 + r$ columns. For example, if $\lambda = (4, 2, 2, 1)$ then $r = 4$ and $\lambda_1 + r = 8$.
Take the path sequence of $\lambda$ and the empty partition:
Place the path sequence of \( I \) at the bottom of the grid and the path sequence of \( \emptyset \) at the top of the grid.
Finally place only right arrows along the horizontal boundary:
Theorem

Denote the partition function with these boundary conditions \( Z_\lambda \). Then \( Z_\lambda \) is equal to the Schur function \( s_\lambda \) for any partition \( \lambda \).

Idea of Proof: Construct a weight preserving bijection between semistandard Young tableaux of shape \( \lambda \) and fillings of the lattice with the boundary conditions just described.
Example:
Write the tableaux as a sequence of partitions:
Take the path sequence of each partition:
Start with 4 by 8 lattice as before:
Add path sequence of empty partition to the top:
Add path sequence of second partition to the next row:
Continue:
Then there is only one possible admissible state with this choice of up arrows.
Ribbon Tableaux

- **n-ribbon**: Some skew-shape containing \( n \) "unit boxes" without a \( 2 \times 2 \) square.

  - e.g. 3-ribbon:
    - 4-ribbon:

- **Semi-standard n-ribbon Tableaux**
  - Tile a Young diagram with \( n \)-ribbons
  - then fill each ribbon with numbers like a SSYT

  - The part with same number forms a horizontal strip (define later)
E.g. \[
\begin{array}{c}
1 & 2 \\
4 & 2 \\
3 & 4
\end{array}
\in RT_3^4(\lambda) \quad \lambda = 5, 4, 4, 1, 1
\]

E.g. \[
\begin{array}{c}
1 & 2 \\
2 & 2 \\
2 & 2
\end{array}
\]

is not a Young diagram.

Indeed, NO RT for this diagram.

\[
\begin{array}{c}
1 & 2 \\
2 & 2 \\
2 & 2
\end{array}
\]

is not a horizontal strip.
A horizontal strip is a collection of ribbons which forms a skewed shape, such that:

- The upper right box of each ribbon has to touch the air, i.e., nothing above it.

E.g.:

![Diagram](image-url)
\textbf{Spin} 🌟

- The spin of a ribbon is height $-1$.
  
  e.g. $\text{spin}\left( \begin{array}{c} \hline \end{array} \right) = 2$

- The spin of a ribbon tableau is sum of spin.
  
  e.g. $\text{spin}\left( \begin{array}{c} 1 \hline 1 \\hline 2 \end{array} \right) = 1 + 1 + 2 = 4$
Ribbon Function (Lascoux, Leclerc & Thibon)

- Let $\lambda / \mu$ be a skew partition tilable by $n$-ribbons.

$$G_{\lambda / \mu}^{n}(x, q) = \sum_{T} q^{\text{spin}(T)} w^{T}(T)$$

\[ \begin{array}{cccc}
1 & 1 & 1 & \\
2 & 2 & & \\
3 & & & \\
\end{array} \]

\[ \Rightarrow q^{3} x_{1}^{2} x_{2}^{2} x_{3} \]

(Thus) Ribbon Fns are Symmetric
Ribbon Lattice Models for \( n \)-ribbon fin.

\[
\begin{array}{c}
\text{\{n, with (n+1) in-arrow (out-arrow)} \\
\end{array}
\]

Think of the vertex as:

\[
\text{...}
\]

... then
The weight of Ribbon vertex

1. Don't allow changing arrow on straight edge:
   eg. $\delta = 1$ if , $\delta = 0$ if

2. $\delta(v) = 1$ if a left arrow entering through bended edge

\[
\delta(v) = \sum_{i} x_i \text{ if in the } i\text{th row}
\]
3. The spin \( \star \)

i) \[ \text{Diagram} \]

ii) \[ \text{Diagram} \]

iii) \[ \text{Diagram} \]

iv) \[ \text{Diagram} \]

\[ \sigma(v) = \# \text{ of} \ < \ \text{in} \]
From Ribbon Tableaux to Lattice model.

- Think of Ribbon Tableaux as sequence of partitions.

\[ \begin{array}{c}
1 \downarrow 2 \downarrow 3 \\
\Rightarrow \emptyset \subset \begin{array}{c}
\lambda_0 \\
\lambda_1 \\
\lambda_2 \\
\lambda_3
\end{array}
\end{array} \]

- Same boundary condition
$q^4 x_1$
$q^2 x_2^2$
$q^3 x_3^2$
peeling off one n-ribbon (if time)

i) numbering the two edge sequence (blue red dots) from 0 to n

ii) The n-th • is moved to the 0-th •, everything else stays.

iii) in the lattice

iv) # intersection = spin
peeling off one horizontal ribbon strip. (if time)

- b/c the top-right box of each ribbon has nothing above it.

we can glue small ribbons up to make the entire ship.

e.g.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{exampleDiagram}
\end{figure}
Yang–Baxter Equation (a.k.a. star–triangle equality)

want a new set of vertex \( \times \) \( \circ \) ... with certain weight.

Such that

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (1,0) -- (0,-1) -- cycle;
\draw[red] (0,0) -- (1,1);
\draw[red] (0,-1) -- (1,0);
\draw[blue] (0,0) -- (0,-1);
\draw[blue] (1,1) -- (1,0);
\end{tikzpicture}} = \begin{tikzpicture}
\draw (0,0) -- (1,1) -- (1,0) -- (0,-1) -- cycle;
\draw[red] (0,0) -- (1,1);
\draw[red] (0,-1) -- (1,0);
\draw[blue] (0,0) -- (0,-1);
\draw[blue] (1,1) -- (1,0);
\end{tikzpicture}
\end{array}
\]

\[\sum (\text{wt(LHS)}) = \sum (\text{wt(RHS)})\] for all boundary

star–triangle \( \times \) \( \circ \) for larger ribbon looks like:

\[
\begin{array}{c}
\text{\begin{tikzpicture}
\draw (0,0) -- (1,1) -- (1,0) -- (0,-1) -- cycle;
\draw[red] (0,0) -- (1,1);
\draw[red] (0,-1) -- (1,0);
\draw[blue] (0,0) -- (0,-1);
\draw[blue] (1,1) -- (1,0);
\end{tikzpicture}} = \begin{tikzpicture}
\draw (0,0) -- (1,1) -- (1,0) -- (0,-1) -- cycle;
\draw[red] (0,0) -- (1,1);
\draw[red] (0,-1) -- (1,0);
\draw[blue] (0,0) -- (0,-1);
\draw[blue] (1,1) -- (1,0);
\end{tikzpicture}
\end{array}
\]
• We conjecture that our lattice model is solvable, i.e., there exist YBEs.

• The YBE for 1, 2, 3-ribbon lattice is computed via SAGE.
Application of the Lattice model.

We can derive various identities of Ribbon $F_n$ using our lattice.

E.g. dual Cauchy identity

Prime rule

$q=1$ ribbon fn is product of Schur fn's.
Thank You!