Extended Nestohedra and their Face Numbers

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UMN REU

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Nestohedra are a well-understood class of convex polytopes

Generalized by Lam–Pylyavskyy ’15 and Devadoss–Heath–Vipismakul ’11 independently

- LP-algebras
- Moduli space of a Riemann surface
What is known so far

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How are they related?

Goal: fill in the column!
Building Sets

Definition

A (connected) building set $\mathcal{B}$ on $[n] := \{1, \ldots, n\}$ is a collection of subsets of $[n]$ such that

1. $\mathcal{B}$ contains all singletons $\{i\}$ and the whole set $[n]$.
2. If $I, J \in \mathcal{B}$ with $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$.

Definition

For an undirected graph $G$, its corresponding graphical building set $\mathcal{B}_G$ is

$$\mathcal{B}_G = \{ I \subseteq V(G) \mid G[I] \text{ is connected}\}.$$
Examples of Building Sets

Complete graph $K_n$
- all subsets of $[n]$
- $\mathcal{B}_{K_4} = \{1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 234, 124, 134, 1234\}$

Path graph $P_n$
- all interval subsets of $[n]$
- $\mathcal{B}_{P_3} = \{1, 2, 3, 12, 23, 123\}$

Star graph $K_{1,n}$
- All singletons and all subsets of $[n + 1]$ that contain $n + 1$
- $\mathcal{B}_{K_{1,3}} = \{1, 2, 3, 4, 14, 24, 34, 124, 134, 234, 1234\}$
Nested Collections

Definition

For a building set $B$, a nested collection $N$ of $B$ is a collection of elements $\{I_1, \ldots, I_m\}$ of $B \setminus \{[n]\}$ such that

1. for any $i \neq j$, $I_i$ and $I_j$ are either nested or disjoint
2. for any $I_{i_1}, \ldots, I_{i_k}$ pairwise disjoint, their union is not an element of $B$

Consider $B = B_{P_4} = \{1, 2, 3, 4, 12, 23, 34, 123, 234, 1234\}$.

- $\{1, 3, 34\}$ is a nested collection
- $\{1, 2, 23\}$ is not a nested collection since $\{1\} \cup \{2\} \in B$. 
Nested Complexes

Definition

For a connected building set $B$ on $[n]$, the **nested set complex** $N(B)$ is the simplicial complex with
- vertices $\{ l \mid l \in B \setminus [n] \}$
- faces $\{ l_1, \ldots, l_m \}$ that are nested collections of $B$

Definition

The **nestohedron** $P(B)$ is the polytope dual to the nested set complex $N(B)$.

In the literature, $P(B_{P_n})$ is known as the **associahedron**, and $P(B_{K_n})$ is known as the **permutohedron**.
Extended Nested Collections

Definition

For a building set $B$ on $[n]$, an extended nested collection $N^\square$ of $B$ is a collection of elements $\{l_1, \ldots, l_m, x_{i_1}, \ldots, x_{i_r}\}$ such that

1. $l_k \in B$ for all $k$, and $\{l_1, \ldots, l_m\}$ form a nested collection of $B$
2. $i_j \in [n]$ for all $j$, and $i_j \not\in l_k$ for all $1 \leq k \leq m$

$B = B_{P_4}$

- $\{1, 3, 34, x_2\}$ is an extended nested collection
- $\{1, 3, 34, x_4\}$ is not an extended nested collection
**Extended Nested Complexes and Nestohedra**

**Definition**

For a building set $\mathcal{B}$ on $[n]$, the extended nested set complex $\mathcal{N}(\mathcal{B})$ is the simplicial complex with

- vertices $\{I | I \in \mathcal{B}\} \cup \{x_i | i \in [n]\}$
- faces $\{I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r}\}$ that are extended nested collections of $\mathcal{B}$

$$\mathcal{B} = \{1, 2, 3, 12, 23, 123\}$$
Definition

For a building set $\mathcal{B}$ on $[n]$, the extended nested set complex $\mathcal{N} □ (\mathcal{B})$ is the simplicial complex with

- vertices $\{ I \mid I \in \mathcal{B} \} \cup \{ x_i \mid i \in [n] \}$
- faces $\{ I_1, \ldots, I_m, x_{i_1}, \ldots, x_{i_r} \}$ that are extended nested collections of $\mathcal{B}$

Definition

The extended nestohedron $\mathcal{P} □ (\mathcal{B})$ is the polytope dual to the extended nested set complex
## What is known so far

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Theorem (Manneville – Pilaud ’17)

Let $G$, $G'$ be undirected graphs such that $\mathcal{N}^\square(B_G) \simeq \mathcal{N}(B_{G'})$. Then $G$ is a spider and $G'$ is the corresponding octopus.
When is $\mathcal{N}(\mathcal{B}) \cong \mathcal{N}(\mathcal{B}')$?

**Theorem (Manneville–Pilaud ’17)**

Let $G, G'$ be undirected graphs such that $\mathcal{N}(\mathcal{B}_G) \cong \mathcal{N}(\mathcal{B}_{G'})$. Then $G$ is a spider and $G'$ is the octopus.
When is $\mathcal{N}^{\Box}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$?

**Corollary (Manneville–Pilaud ’17)**

- $\mathcal{N}^{\Box}(\mathcal{B}_K) \simeq \mathcal{N}(\mathcal{B}_{K_{1,n}})$ is the dual of the stellohedron.
- $\mathcal{N}^{\Box}(\mathcal{B}_P) \simeq \mathcal{N}(\mathcal{B}_{P_{n+1}})$ is the dual of the $(n-2)$-associahedron.

**Remark (REU ’19)**

When $G = C_4$, we do not have $\mathcal{N}^{\Box}(\mathcal{B}_G) \simeq \mathcal{N}(\mathcal{B}')$ for any other building set $\mathcal{B}'$.

**Theorem (REU ’19)**

If $\mathcal{B}$ is a building set on $[n]$ such that all elements $I \in \mathcal{B}$ are intervals, then there exists $\mathcal{B}'$ such that $\mathcal{N}^{\Box}(\mathcal{B}) \simeq \mathcal{N}(\mathcal{B}')$. 
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When is $\mathcal{N}^\square(\mathcal{B})$ flag?

**Definition**

A simplicial complex $\Delta$ is **flag** if $\Delta$ has no minimal non-faces of degree greater than 2. In other words, $\Delta$ is determined by its 1-skeleton.

**Proposition (REU ’19)**

$\mathcal{N}(\mathcal{B})$ is flag if and only if $\mathcal{N}^\square(\mathcal{B})$ is flag.

For a graphical building set $\mathcal{B} = \mathcal{B}_G$, it was shown in (PRW ’08) that $\mathcal{N}(\mathcal{B})$ is a flag simplicial complex.

**Corollary (REU ’19)**

If $G$ is an undirected graph, then $\mathcal{N}^\square(\mathcal{B}_G)$ is flag.
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Link Decompositions of $\mathcal{N}(\mathcal{B})$ and $\mathcal{N}^\square(\mathcal{B})$

**Theorem (Zelevinsky ’06)**

Let $\mathcal{B}$ be a building set on $S$. Then the link of $C \in \mathcal{B}$ in $\mathcal{N}(\mathcal{B})$

$$\mathcal{N}(\mathcal{B})_C \simeq \mathcal{N}(\mathcal{B}|_C) \ast \mathcal{N}(\mathcal{B}/C).$$

**Theorem (REU ’19)**

For the extended nested complex $\mathcal{N}^\square(\mathcal{B})$, we have:

$$\mathcal{N}^\square(\mathcal{B})_{x_i} \simeq \mathcal{N}^\square(\mathcal{B}_1) \ast \cdots \ast \mathcal{N}^\square(\mathcal{B}_k)$$

where $\mathcal{B}_1, \ldots, \mathcal{B}_k$ are the connected components of $\mathcal{B}|_{[n]\setminus\{i\}}$, and

$$\mathcal{N}^\square(\mathcal{B})_C \simeq \mathcal{N}(\mathcal{B}|_C) \ast \mathcal{N}^\square(\mathcal{B}/C)$$

for $C \in \mathcal{B}$. 
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Polytopality

Theorem (REU ’19)

For any building set $B$, $\mathcal{N}_{\Box}(B)$ can be realized as the boundary of a polytope $\mathcal{N}_B$. 
Consider $\mathbb{R}^n$ with standard basis vectors $e_1, \ldots, e_n$. Start with cross polytope in $\mathbb{R}^n$ with vertices $e_i$ labeled $\{i\} \in \mathcal{B}$ and vertices $-e_i$ labeled $x_i$ for all $i \in [n]$.

$\mathcal{B} = \{1, 2, 3, 12, 123\}$
Polytopality

- Consider $\mathbb{R}^n$ with standard basis vectors $e_1, \ldots, e_n$. Start with cross polytope in $\mathbb{R}^n$ with vertices $e_i$ labeled $\{i\} \in \mathcal{B}$ and vertices $-e_i$ labeled $x_i$ for all $i \in [n]$.

- Order the non-singletons of $\mathcal{B}$ by decreasing cardinality, then for each $I \in \mathcal{B}$ a non-singleton, perform stellar subdivision on the face $\mathcal{I} = \{\{i\} \mid i \in I\}$, with the new added vertex labeled $I$.

\[
\mathcal{B} = \{1, 2, 3, 12, 123\}
\]
Consider $\mathbb{R}^n$ with standard basis vectors $e_1, \ldots, e_n$. Start with cross polytope in $\mathbb{R}^n$ with vertices $e_i$ labeled $\{i\} \in \mathcal{B}$ and vertices $-e_i$ labeled $x_i$ for all $i \in [n]$.

Order the non-singletons of $\mathcal{B}$ by decreasing cardinality, then for each $l \in \mathcal{B}$ a non-singleton, perform stellar subdivision on the face $I = \{\{i\} \mid i \in l\}$, with the new added vertex labeled $l$.

The boundary of the resulting polytope $\mathcal{N}_\mathcal{B}$ will be isomorphic to $\mathcal{N}^\square(\mathcal{B})$.

$$\mathcal{B} = \{1, 2, 3, 12, 123\}$$
We also obtain a polytopal realization of \( \mathcal{P}^\square(B) \) as a Minkowski sum.

**Theorem (REU ’19)**

Let \( B \) a building set on \( [n] \), and consider \( \mathbb{R}^n \) with standard basis vectors \( e_1, \ldots, e_n \). Then \( \mathcal{P}^\square(B) \) is isomorphic to the boundary of the polytope:

\[
\mathcal{P} := \sum_{i \in [n]} \text{Conv}(0, e_i) + \sum_{I \in B} \text{Conv}\{e_S | S \subsetneq I\},
\]

where the coordinates of \( e_S \) are given by the indicator function on \( S \) i.e. \((e_S)_i = 1\) if and only if \( i \in S \).

Intuitively, the first sum is the \( n \)-dimensional cube \( C^n \), while each term of the next sum corresponds to shaving a face \( I \in B \) from the cube.
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Definition

For a polytope $\mathcal{P}$, let $f_k$ be the number of $k$-dimensional faces of $\mathcal{P}$. The $f$-vector of $\mathcal{P}$ is defined to be $f = (f_{-1}, \ldots, f_{d-1})$.

Definition

The $h$-vector $h = (h_0, \ldots, h_d)$ of $\mathcal{P}$ is defined by

$$
\sum_{i=0}^{d} h_i t^i = \sum_{i=0}^{d} f_{i-1}(t - 1)^{i-1}
$$

If $\mathcal{P}$ is a simple polytope, then we have $h_i = h_{d-i}$ for all $i = 0, \ldots, \lfloor \frac{d}{2} \rfloor$. 

Proposition (REU ’19)

\[ f_{\mathcal{P}^\Box(B)}(t) = \sum_{S \subseteq [n]} (t + 1)^{n-|S|} f_{\mathcal{P}(B|S)}(t) \]
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Gal’s Conjecture for Flag $\mathcal{P}^\square(\mathcal{B})$

**Definition**

The $\gamma$-vector for a simple polytope $\mathcal{P}$ is given by

$$ \gamma_i t^i (t + 1)^{d-2i} = \sum_{j=0}^{d} h_j t^j. $$

**Gal’s Conjecture (2005)**

The $\gamma$-vector of any flag simple polytope is nonnegative.

- Shown true for $\mathcal{P}(\mathcal{B})$ by Volodin ’10
Gal’s Conjecture for Flag $\mathcal{P}^\square(\mathcal{B})$

Theorem (REU ’19)
Gal’s conjecture is true for flag extended nestohedra $\mathcal{P}^\square(\mathcal{B})$.

- Start with flag building set $\mathcal{B}$
- There exists minimal flag building set $\mathcal{B}_{\text{min}} \subseteq \mathcal{B}$, and $\mathcal{P}^\square(\mathcal{B}_{\text{min}})$ has nonnegative $\gamma$-vector
- Add back in elements $\mathcal{B} \setminus \mathcal{B}_{\text{min}}$
  - Corresponds to shaving a codimension 2 face
- Use link decomposition to show that $\gamma$-vector remains nonnegative
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Gal’s Conjecture for Flag $\mathcal{P}^\square(B)$

- **Chordal**: nice class of building sets, includes $\mathcal{B}_{K_n}, \mathcal{B}_{P_n}, \mathcal{P}_{K_{1,n}}$
- $\hat{\mathcal{S}}_n(B) = \{B\text{-permutations with no double or final descents}\}$

**Theorem (Postnikov–Reiner–Williams ’08)**

For chordal $B$ on $[n]$, 
$$\gamma_{\mathcal{P}(B)}(t) = \sum_{w \in \hat{\mathcal{S}}_n(B)} t^{\text{des}(w)}.$$

- $\hat{\mathcal{S}}_{n+1} = \{\text{extended } B\text{-permutations with no double or final descents}\}$

**Theorem (REU ’19)**

For chordal $B$ on $[n]$, 
$$\gamma_{\mathcal{P}^\square(B)}(t) = \sum_{w \in \hat{\mathcal{S}}_{n+1}(B)} t^{\text{des}(w)}.$$
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Weak Bruhat Order

- $w = (a_1 \ a_2 \ \cdots \ a_n) \in S_n$
- Transpositions $s_i = (i \ i + 1)$
- $\ell(w) := |\{1 \leq i < j \leq n \mid a_i > a_j\}|$, i.e. the minimum number of transpositions

Definition

The **weak Bruhat order** on $S_n$ is defined by the following:

$\pi \triangleleft \sigma$ if and only if $\ell(\sigma) = \ell(\pi) + 1$ and $\sigma = \pi \cdot s_i$
Weak Bruhat Order

weak Bruhat order on $S_3$  inversion sets
Partial Permutations

**Definition**

Define the set of **partial permutations** on \([n]\), denoted \(\mathcal{P}_n\), to be set of permutations \(w \in \mathcal{S}_S\) for some \(S \subseteq [n]\).

\[\mathcal{P}_2 : \begin{array}{c}
(1 \ 2), \ (2 \ 1), \ (1), \ (2), \ () \\
S=\{1,2\} \quad S=\{1\} \quad S=\{2\} \quad S=\emptyset
\end{array}\]

**Remark**

- \(\mathcal{S}_n\) is in bijection with facets of \(\mathcal{N}(\mathcal{B}_{K_n})\)
- \(\mathcal{P}_n\) is in bijection with the facets of \(\mathcal{N}(\mathcal{B}_{K_n})\)
Partial Order on $\mathcal{P}_n$

**Definition (REU ’19)**

Define map $\varphi : \mathcal{P}_n \rightarrow S_{n+1}$ as follows.

- Consider partial permutation $w \in S_S$, $S \subseteq [n]$
- Append numbers in $[n+1] \setminus S$ to end of $w$ in descending order
- Resulting permutation $\varphi(w) \in S_{n+1}$

Let $w = (2 \ 4 \ 1) \in \mathcal{P}_5 \implies \varphi(w) = (2 \ 4 \ 1 \ 6 \ 5 \ 3)$

**Definition (REU ’19)**

The **partial order** on $\mathcal{P}_n$ defined by the following:

$\pi < \sigma$ if and only if $\varphi(\pi) < \varphi(\sigma)$ in the weak Bruhat order on $S_{n+1}$
Partial Order on $\mathfrak{S}_n$

\[ \begin{array}{c}
\emptyset \\
(2) \\
(2 1) \\
(1 2) \\
(1 2 3) \\
(2 1 3) \\
(2 3 1) \\
(1 3 2) \\
(1 3 2 1) \\
(3 2 1) \\
\end{array} \]

$\mathfrak{S}_2$ weak Bruhat order on $\varphi(\mathfrak{S}_2)$
Definition

A **congruence** on a lattice $L$ is an equivalence relation $\Theta$ on elements of $L$ which respects joins and meets, i.e. if $a_1 \equiv a_2$ and $b_1 \equiv b_2$, then

$$a_1 \land b_1 \equiv a_2 \land b_2,$$
$$a_1 \lor b_1 \equiv a_2 \lor b_2.$$  

A **lattice quotient** $L/\Theta$ is a partial order on the equivalence classes under $\Theta$:

$$[a]_\Theta \leq [b]_\Theta \iff \exists x \leq_L y \text{ for some } x \in [a], y \in [b].$$

Proposition (REU '19)

The defined partial order on $\mathfrak{S}_n$ is a lattice quotient of the weak Bruhat order on $\mathfrak{S}_{n+1}$. 
Corollary (McConville ’16, Reading ’02)

- Every interval of $\mathcal{P}_n$ is contractible or homotopy equivalent to a sphere
- If $x = \vee_{\mathcal{P}_n} Y$ for some $Y \subseteq \mathcal{P}_n$, then $x = \vee_{\mathcal{S}_{n+1}} Y$
- Möbius function $\mu(u, v)$ only takes on values 0, $\pm 1$
- **Shellings**: nice way to build up a simplicial complex facet by facet

**Theorem (Björner ’84)**

Label facets of $\mathcal{N}(\mathcal{B}_{K_n})$ by permutations $w \in S_n$. If $\pi_1 < \cdots < \pi_k$ is a linear extension of the weak Bruhat order on $S_n$, then $F_{\pi_1}, \ldots, F_{\pi_k}$ gives a shelling of $\mathcal{N}(\mathcal{B}_{K_n})$.

**Theorem (REU ’19)**

Label facets of $\mathcal{N}(\mathcal{B}_{K_n})$ by partial permutations $w \in \mathcal{P}_n$. If $\pi_1 < \cdots < \pi_k$ is a linear extension of the partial order on $\mathcal{P}_n$, then $F_{\pi_1}, \ldots, F_{\pi_k}$ gives a shelling of $\mathcal{N}(\mathcal{B}_{K_n})$. 
<table>
<thead>
<tr>
<th>What is known so far</th>
<th>Non-extended</th>
<th>Extended ($\Box$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>When flag</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Link decomposition</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Polytopality</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Gal’s conjecture</td>
<td>Y</td>
<td>Y</td>
</tr>
<tr>
<td>Combinatorial interpretation for $\gamma$-vector</td>
<td>chordal $B$</td>
<td>chordal $B$</td>
</tr>
<tr>
<td>Shellings</td>
<td>$B_{K_n}$</td>
<td>$B_{K_n}$</td>
</tr>
<tr>
<td>Cluster/LP algebras</td>
<td>Y</td>
<td>?</td>
</tr>
<tr>
<td>How are they related?</td>
<td>$\mathcal{N}(B) \simeq \mathcal{N}(B')$ sometimes, through $f$- and $h$-vectors, ...?</td>
<td></td>
</tr>
</tbody>
</table>
Future Work

- Is there a combinatorial interpretation for the $\gamma$-vector of $\mathcal{P}(\mathcal{B})$, $\mathcal{P}^\Box(\mathcal{B})$ of arbitrary flag building sets?
- When does a total ordering on (extended) $\mathcal{B}$-permutations give a shelling of the (extended) nested complexes?
- Can $\mathcal{N}^\Box(\mathcal{B})$ provide a combinatorial interpretation of the exchange polynomials of LP-algebras? (Lam–Pylyavskyy)

Conjecture

Let $G$ be a forest and $L(G)$ be the line graph of $G$. Then

$$f_{\mathcal{P}(\mathcal{B}_G)}(t) = f_{\mathcal{P}^\Box(\mathcal{B}_{L(G)})}(t).$$
Acknowledgements and References

- Thank you to Vic Reiner and Sarah Brauner for all of their support and guidance!
- See our REU report for a complete set of references
Questions?

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