Lattice Models, Differential Forms, and the Yang-Baxter Equation

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What is a Lattice Model?
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- Origins in statistical mechanics, studied by Baxter [1].
- Grid with labeled edges.
- Labelings around a vertex locally satisfy some property.
Six-Vertex Model
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Observation: A state $g_{i,j}$, $f_{i,j}$, $g_{i,j+1}$, $f_{i,j+1}$ is admissible iff $f_{i,j+1} - f_{i,j} \equiv g_{i+1,j} - g_{i,j} \pmod{3}$.
\[ f_{i,j+1} - f_{i,j} \equiv g_{i+1,j} - g_{i,j} \pmod{3} \]
\[ \iff D_y f = D_x g \]
\[ \iff fdx + gdy \text{ is closed.} \]

- \( f \) and \( g \) are functions on a rectangular grid, take values in \( \mathbb{F}_3 \).
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$\iff f dx + g dy$ is closed.

- $f$ and $g$ are functions on a rectangular grid, take values in $\mathbb{F}_3$.
- Admissible 1-form $f dx + g dy$: $f$ and $g$ only equal 0 and 1.
Differential Forms

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\[ \iff f \, dx + g \, dy \text{ is closed.} \]

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- Admissible 1-form \( f \, dx + g \, dy \): \( f \) and \( g \) only equal 0 and 1.
- So admissible states \( \iff \) closed admissible 1-forms.
Exterior derivative: for $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{F}_3$,

$$dh := (D_x h)dx + (D_y h)dy.$$
Differential Forms

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$$dh := (D_x h)dx + (D_y h)dy.$$ 

- A 1-form $\alpha$ is exact if $\alpha = dh$ for some function $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{F}_3$. 

Idea: Every closed 1-form on an open ball is exact, so same should be true for a discrete grid.

Lemma: Every closed 1-form on $\{1, 2, \ldots, m\} \times \{1, 2, \ldots, n\}$ is exact.
Differential Forms

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Lemma

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3-Colorings

- We have a correspondence

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\{\text{Closed 1-forms}\} \leftrightarrow \{\text{Functions}\} \times \{\text{Initial condition}\}
\]

given by

\[h \leftrightarrow (dh, h_0).\]
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Using this correspondence, we can prove

**Theorem**

*We have a one-to-one correspondence*

\[ \{\text{Admissible states}\} \leftrightarrow \{\text{3-colorings of a rectangular grid}\} \times \mathbb{F}_3. \]
Toroidal Boundary Conditions
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- Same treatment as before - discrete differential forms.

Lemma

Every closed 1-form on the discrete torus can be written uniquely in the form $rdx + sdy + \omega$, where $r, s \in F^3$ and $\omega$ is exact.
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Lemma

Every closed 1-form on the discrete torus can be written uniquely in the form

$$r dx + s dy + \omega,$$

where $r, s \in \mathbb{F}_3$ and $\omega$ is exact.
3-colorings of a rectangular grid $\leftrightarrow$ functions $h$ such that $D_x h, D_y h \neq 0$, and $h_{1,1} = 0$. 

Call $h$ sparse if neither $D_x h$ nor $D_y h$ are surjective, and $h_{1,1} = 0$. 

No nice correspondence with 3-colorings in toroidal case, but we have

Theorem 

There is a one-to-one correspondence between sparse functions and admissible states of the six-vertex model with toroidal boundary conditions.
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3-colorings of a rectangular grid ↔ functions $h$ such that $D_x h, D_y h \neq 0$, and $h_{1,1} = 0$.

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**Theorem**

*There is a one-to-one correspondence between sparse functions and admissible states of the six-vertex model with toroidal boundary conditions.*
Eight-Vertex Model

- Observation: A state $g_{i,j}, g_{i,j+1}, f_{i,j+1}, f_{i,j}$ is admissible iff

$$f_{i,j+1} - f_{i,j} \equiv g_{i+1,j} - g_{i,j} \pmod{2}.$$
Eight-Vertex Model

We *could* use differential calculus again, but there is an easier approach.
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- Set of admissible states is a vector space over $\mathbb{F}_2$. 
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- Everything is a linear condition.
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Set of admissible states is a vector space over $\mathbb{F}_2$.

Everything is a linear condition.

Easy to count the number of admissible states.

**Theorem**

The number of admissible states of the eight-vertex model is $2^{m+n+mn}$. 
Question: Given a set of boundary conditions, how many admissible states do they have?
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By linear algebra, this essentially does not depend on what the boundary conditions are.

Admissible states of “homogeneous lattice” $\leftrightarrow$ Admissible states of lattice with given boundary conditions.

$$L_0 \leftrightarrow L_B + L_0$$
New question: when does a set of boundary conditions have an admissible state?
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Answer: when the boundary values sum to 0.

**Theorem**

Let $B$ be a set of boundary values that sum to 0. Then the number of admissible states with boundary conditions $B$ is $2^{(m-1)(n-1)}$. 
Adding Weights

$$\begin{array}{cccc}
\begin{array}{cc}
0 & 1 \\
0 & 1 \\
0 & 1 \\
\end{array}
\end{array}
\begin{array}{cc}
1 & 1 \\
1 & 1 \\
1 & 1 \\
\end{array}
\begin{array}{cc}
1 & 0 \\
0 & 0 \\
1 & 0 \\
\end{array}
\begin{array}{cc}
0 & 0 \\
0 & 0 \\
1 & 1 \\
\end{array}
\begin{array}{cc}
a_1 & a_{-1} \\
b_1 & b_{-1} \\
c_1 & c_{-1} \\
d_1 & d_{-1} \\
\end{array}$$
Adding Weights

\[
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
1 & 1 & 0 & 0 \\
\end{array}
\]

\[
\begin{array}{cccc}
a_1 & & \cdot & \cdot \\
& & & \\
\cdot & & & \cdot \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
a_{-1} & & \cdot & \cdot \\
& & & \\
\cdot & & & \cdot \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
b_1 & & \cdot & \cdot \\
& & & \\
\cdot & & & \cdot \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
b_{-1} & & \cdot & \cdot \\
& & & \\
\cdot & & & \cdot \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
c_1 & & \cdot & \cdot \\
& & & \\
\cdot & & & \cdot \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
c_{-1} & & \cdot & \cdot \\
& & & \\
\cdot & & & \cdot \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
d_1 & & \cdot & \cdot \\
& & & \\
\cdot & & & \cdot \\
& & & \\
\end{array}
\]

\[
\begin{array}{cccc}
d_{-1} & & \cdot & \cdot \\
& & & \\
\cdot & & & \cdot \\
& & & \\
\end{array}
\]
Yang-Baxter Equation

\[ \sum_{\gamma, \mu, \nu} \alpha \beta \gamma \delta \theta \rho \sigma \tau = \sum_{\delta, \phi, \psi} \alpha \beta \gamma \delta \theta \rho \sigma \tau \]
Question: Given $S$ and $T$, when does there exist (nontrivial) $R$ such that YBE holds?
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Galleas and Martins [2] answered this question in the case $c_1 = c_{-1}$ and $d_1 = d_{-1}$. 
Question: Given $S$ and $T$, when does there exist (nontrivial) $R$ such that YBE holds?

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YBE can be expressed as a matrix equation

$$R_{12}S_{13}T_{23} - T_{23}S_{13}R_{12} = 0.$$
Explicit Computations

\[
\begin{align*}
a_j(T)a_j(S)d_i(R) + d_i(T)c_i(S)a_{-j}(R) &= c_i(T)d_i(S)a_j(R) + b_{-j}(T)b_{-j}(S)d_i(R) \\
d_i(T)b_j(S)c_i(R) + a_j(T)d_i(S)b_{-j}(R) &= b_j(T)d_i(S)a_j(R) + c_{-i}(T)b_{-j}(S)d_i(R) \\
d_i(T)b_j(S)b_j(R) + a_j(T)d_i(S)c_{-i}(R) &= d_i(T)a_j(S)a_j(R) + a_{-j}(T)c_{-i}(S)d_i(R) \\
c_i(T)a_j(S)c_i(R) + b_j(T)c_i(S)b_{-j}(R) &= a_j(T)c_i(S)a_j(R) + d_{-i}(T)a_{-j}(S)d_i(R) \\
c_i(T)a_j(S)b_j(R) + b_j(T)c_i(S)c_{-i}(R) &= c_i(T)b_j(S)a_j(R) + b_{-j}(T)d_{-i}(S)d_i(R) \\
b_{-j}(T)a_j(S)c_i(R) + c_{-i}(T)c_i(S)b_{-j}(R) &= d_{-i}(T)d_i(S)b_j(R) + a_j(T)b_{-j}(S)c_i(R) \\
c_1(T)c_{-1}(S)c_1(R) &= c_{-1}(T)c_1(S)c_{-1}(R) \\
d_1(T)c_1(S)d_{-1}(R) &= d_{-1}(T)c_{-1}(S)d_1(R) \\
c_1(T)d_1(S)d_{-1}(R) &= c_{-1}(T)d_{-1}(S)d_1(R) \\
d_1(T)d_{-1}(S)c_1(R) &= d_{-1}(T)d_1(S)c_{-1}(R)
\end{align*}
\]
Necessary Conditions

Theorem

**Necessary conditions for a solution with** \( c_{-1}(R), c_1(R), d_{-1}(R), d_1(R) \) **nonzero include**

\[
\begin{align*}
a_1(T)b_1(T)F(S) &= a_{-1}(T)b_{-1}(T)F(S) \\
a_1(S)b_1(S)F(T) &= a_{-1}(S)b_{-1}(S)F(T) \\
\frac{c_i(T)d_{-i}(T)}{c_{-i}(T)d_i(T)}G_i(S, T)^2 &= [a_1(T)b_1(T)F(S) - a_1(S)b_1(S)F(T)]^2 \\
\frac{c_1(T)c_{-1}(S)}{c_{-1}(T)c_1(S)} &= \frac{d_1(T)d_{-1}(S)}{d_{-1}(T)d_1(S)}.\end{align*}
\]
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References

Exactly Solved Models in Statistical Mechanics.

Yang-Baxter equation for the asymmetric eight-vertex model.
Physical review E, 11.