Formulas for Birational R-Matrix Action

Sunita Chepuri, Feiyang Lin*
TA: Emily Tibor

UMN Combinatorics REU 2020

August 7, 2020
The Birational R-Matrix, $\eta$

Why we care:

- Relates to networks on a cylinder;
- Describes relations between matrix factorizations;
- Occurs in the study of geometric crystals;
- The tropicalization is the combinatorial R-matrix of affine crystals;
- Has applications to discrete Painlevé dynamical systems.
The Birational R-Matrix, $\eta$

Let $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ be two sets of formal variables, where $n \geq 1$. For $1 \leq i \leq n$, let

$$\kappa_i(a, b) = \sum_{j=i}^{i+n-1} \prod_{k=j+1}^{i+n-1} b_k \prod_{k=i+1}^{j} a_k,$$

where the indices $k$ are taken mod $n$. Then

$$\eta : (a, b) \mapsto (b', a')$$

where $a' = (a'_1, \ldots, a'_n)$, $b' = (b'_1, \ldots, b'_n)$, and

$$a'_i = \frac{a_{i-1} \kappa_{i-1}(a, b)}{\kappa_i(a, b)}$$

$$b'_i = \frac{b_{i+1} \kappa_{i+1}(a, b)}{\kappa_i(a, b)}.$$
Example of $\eta$

\[
\kappa_i(a, b) = \sum_{j=i}^{i+n-1} \prod_{k=i+1}^{j} b_k \prod_{k=j+1}^{i+n-1} a_k,
\]

\[a_i' = \frac{a_{i-1}\kappa_{i-1}(a, b)}{\kappa_i(a, b)}.\]

For example, for $n = 4$,

\[a_2' = a_1 \frac{\kappa_1(a, b)}{\kappa_2(a, b)} = a_1 \frac{a_2a_3a_4 + b_2a_3a_4 + b_2b_3a_4 + b_2b_3b_4}{a_3a_4a_1 + b_3a_4a_1 + b_3b_4a_1 + b_3b_4b_1}.\]
\( \eta_i \) and its properties

Let \( x_i = (x_i^{(1)}, \ldots, x_i^{(n)}) \). Now for \( 1 \leq i < m \), let

\[
\eta_i(x_1, \ldots, x_m) = (x_1, \ldots, x_{i-1}, \eta(x_i, x_{i+1}), x_{i+2}, \ldots, x_m).
\]

**Theorem 1.** [Lam–Pylyavskyy, 2008]

The birational R-matrix has the following properties:

- \( \eta \) is an involution: \( \eta^2 = 1 \);
- \( \eta \) satisfies the braid relations: for \( 1 \leq i < m - 1 \),

\[
\eta_i \eta_{i+1} \eta_i(x_1, \ldots, x_m) = \eta_{i+1} \eta_i \eta_{i+1}(x_1, \ldots, x_m).
\]

\( \Rightarrow \) Action of \( S_m \) on \( (x_1, \ldots, x_m) \).
Main Problem

To refer to specific variables after applying a permutation $s$, we write $s(x_i^{(r)})$ to denote the $r$-th variable in the resultant $i$-th vector. When indices are in parentheses, they are taken mod $n$.

**Main Problem.** For any $s \in S_m$, $1 \leq i \leq m$ and $1 \leq r \leq n$, we would like to write $s(x_i^{(r)})$ explicitly as a rational function of the original variables.
Let $j > 1$. Write $s_i$ for the transposition switching $i$ and $i + 1$.

- $s$ is shifting by 1: $s = s_{j-1}s_{j-2} \ldots s_i$ and $s = s_is_{i+1} \ldots s_{j-1}$;
- $s$ is a transposition: $s = s_is_{i+1} \ldots s_{j-2}s_{j-1}s_{j-2} \ldots s_i$;
- Combinatorial interpretation of functions that appear.
The $\tau$, $\sigma$, $\bar{\sigma}$ Functions

Let $n$ be a positive integer, $k$ a nonnegative integer, and let $1 \leq r \leq n$. Then $\tau_k^{(r)}$ is defined as follows:

$$\tau_k^{(r)}(x_1, x_2, \ldots, x_m) = \sum_{1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n} x_{i_1}^{(r)} x_{i_2}^{(r-1)} \cdots x_{i_k}^{(r-k+1)}$$

where no index appears more than $n - 1$ times in the sum. The $\sigma$ and $\bar{\sigma}$ functions are defined using $\tau$:

$$\sigma_k^{(r)}(x_1, x_2, \ldots, x_m) = \sum_{i=0}^{k} x_1^{(r)} x_1^{(r-1)} \cdots x_1^{(r-i+1)} \tau_{k-i}^{(r-i)}(x_2, x_3, \ldots, x_m),$$

$$\bar{\sigma}_k^{(r)}(x_1, x_2, \ldots, x_m) = \sum_{i=0}^{k} \tau_{k-i}^{(r)}(x_1, x_2, \ldots, x_{m-1}) x_m^{(r-k+i)} x_m^{(r-k+i-1)} \cdots x_m^{(r-k)}.$$
Let $n = 4$. Write $a = (a_1, \ldots, a_4)$, $b = (b_1, \ldots, b_4)$, $c = (c_1, \ldots, c_4)$ in place of $x_1$, $x_2$, $x_3$. Then

$$
\tau_5^{(3)}(b, c) = b_3 b_2 b_1 c_4 c_3 + b_3 b_2 c_1 c_4 c_3,
$$

$$
\sigma_6^{(4)}(a, b, c) = \tau_6^{(4)}(b, c) + a_4 \tau_5^{(3)}(b, c) + a_4 a_3 \tau_4^{(2)}(b, c) + a_4 a_3 a_2 \tau_3^{(1)}(b, c) + a_4 a_3 a_2 a_1 \tau_2^{(4)}(b, c) + a_4 a_3 a_2 a_1 a_4 \tau_1^{(3)}(b, c) + a_4 a_3 a_2 a_1 a_4 a_3
$$

$$
\bar{\sigma}_6^{(4)}(a, b, c) = \tau_6^{(4)}(a, b) + \tau_5^{(4)}(a, b) c_3 + \tau_4^{(4)}(a, b) c_4 c_3 + \tau_3^{(4)}(a, b) c_1 c_4 c_3 + \tau_2^{(4)}(a, b) c_2 c_1 c_4 c_3 + \tau_1^{(4)}(a, b) c_3 c_2 c_1 c_4 c_3 + c_4 c_3 c_2 c_1 c_4 c_3
$$
1-Shifts

We call permutations of the form $s_{j-1} \ldots s_i$ and $s_i \ldots s_{j-1}$ 1-shifts. For example, when $i = 1, j = 4$, in cycle notation, $s_3 s_2 s_1 = (4321)$ and $s_1 s_2 s_3 = (1234)$.

**Theorem 2** ([Lam–Pylyavskyy, 2010]; [Chepuri–L., 2020+])

\[
s_{j-1} \ldots s_i(x_{j}^{(r)}) = x_i^{(r-j+i)} \frac{\sigma_{(n-1)(j-i)}^{(r-j+i+1)}(x_i, \ldots, x_j)}{\sigma_{(n-1)(j-i)}^{(r-j+i)}(x_i, \ldots, x_j)},
\]

and for $i \leq k < j$,

\[
s_{j-1} \ldots s_i(x_{k}^{(r)}) = \frac{x_{k+1}^{(r+1)} \sigma_{(n-1)(k+1-i)}^{(r-k+i)}(x_i, \ldots, x_{k+1}) \sigma_{(n-1)(k-i)}^{(r-k+i+1)}(x_i, \ldots, x_k)}{\sigma_{(n-1)(k+1-i)}^{(r-k+i+1)}(x_i, \ldots, x_{k+1}) \sigma_{(n-1)(k-i)}^{(r-k+i)}(x_i, \ldots, x_k)}.
\]
1-Shifts

We call permutations of the form $s_{j-1} \ldots s_i$ and $s_i \ldots s_{j-1}$ 1-shifts. For example, when $i = 1, j = 4$, in cycle notation, $s_3s_2s_1 = (4321)$ and $s_1s_2s_3 = (1234)$.

**Theorem 2 (Dual)** [Chepuri–L. 2020+]

$$s_i \ldots s_{j-1}(x_i^{(r)}) = x_j^{(r+j-i)} \frac{\bar{\sigma}^{(r)}_{(n-1)(j-i)}(x_i, \ldots, x_j)}{\bar{\sigma}^{(r-1)}_{(n-1)(j-i)}(x_i, \ldots, x_j)},$$

and for $i < k \leq j$,

$$s_i \ldots s_{j-1}(x_k^{(r)}) = \frac{x_k^{(r-1)}\bar{\sigma}_{(n-1)(k+1)}^{(r-2)}(x_{k-1}, \ldots, x_j)\bar{\sigma}^{(r)}_{(n-1)(j-k)}(x_k, \ldots, x_j)}{\bar{\sigma}_{(n-1)(j-k+1)}^{(r-1)}(x_{k-1}, \ldots, x_j)\bar{\sigma}^{(r-1)}_{(n-1)(j-k)}(x_k, \ldots, x_j)}.$$
Combinatorial Interpretation of $\tau$ Functions

Cylindrical networks $N(n, m)$:

Figure 1: Illustration of $N(n, m)$
Figure 2: Illustration of $N(3, 4)$
Combinatorial Interpretation of $\tau$ Functions

Highway paths:

Figure 3: A non-example and an example of a highway path
Figure 4: All terms in $\tau_3^{(1)}(a, b, c)$ that use only $b$ and $c$
Combinatorial Interpretation of $\sigma$ and $\bar{\sigma}$ Functions

$$\sigma_6^{(4)}(a, b, c) = \tau_6^{(4)}(b, c) + a_4 \tau_5^{(3)}(b, c) + a_4 a_3 \tau_4^{(2)}(b, c) + a_4 a_3 a_2 \tau_3^{(1)}(b, c)$$

$$+ a_4 a_3 a_2 a_1 \tau_2^{(4)}(b, c) + a_4 a_3 a_2 a_1 a_4 \tau_1^{(3)}(b, c) + a_4 a_3 a_2 a_1 a_4 a_3$$

$$= (\tau_6^{(4)}(b, c) + a_4 \tau_5^{(3)}(b, c) + a_4 a_3 \tau_4^{(2)}(b, c) + a_4 a_3 a_2 \tau_3^{(1)}(b, c))$$

$$+ a_4 a_3 a_2 a_1 (\tau_2^{(4)}(b, c) + a_4 \tau_1^{(3)}(b, c) + a_4 a_3)$$

$$= \tau_6^{(4)}(a, b, c) + a_4 a_3 a_2 a_1 \tau_2^{(4)}(b, c)$$
A transposition that switches $i$ and $j$ can be written as $s_is_{i+1} \ldots s_{j-1} \ldots s_{i+1}s_i$. For example, $(14) = s_1s_2s_3s_2s_1 = s_3s_2s_1s_2s_3$. 
The $\Omega$ Functions

For $i \leq k \leq j - 1$, define

$$(k)\Omega^{(r)}_{(n-1)(j-i)}(x_i, \ldots, x_j) = \sum_{\ell=0}^{n-1} \sigma^{(r)}_{(n-1)(k-i)+\ell}(x_i, \ldots, x_k)\bar{\sigma}^{(r+k-i-\ell)}_{(n-1)(j-k)-\ell}(x_{k+1}, \ldots, x_j).$$

Specializes to $\bar{\sigma}$ when $k = i$ and $\sigma$ when $k = j - 1$. Example: $j = 4$, $i = 1$, $k = 2$, $n = 4$,

$$(2)\Omega^{(r)}_{9}(a, \ldots, d) = \sigma^{(r)}_{3}(a, b)\bar{\sigma}^{(r+1)}_{6}(c, d) + \sigma^{(r)}_{4}(a, b)\bar{\sigma}^{(r)}_{5}(c, d)$$

$$+ \sigma^{(r)}_{5}(a, b)\bar{\sigma}^{(r-1)}_{4}(c, d) + \sigma^{(r)}_{6}(a, b)\bar{\sigma}^{(r-2)}_{3}(c, d).$$
Conjecture 1. [Chepuri–L. 2020+] Let \( s = s_i \ldots s_{j-2}s_{j-1}s_{j-2} \ldots s_i \). For \( i < k < j \),

\[
s(x_k^{(r)}) = x_k^{(r)} \frac{(k) \Omega_{(n-1)(j-i)}^{(r-k+i)}(x_i, \ldots, x_j) (k-1) \Omega_{(n-1)(j-i)}^{(r-k+i-1)}(x_i, \ldots, x_j)}{(k-1) \Omega_{(n-1)(j-i)}^{(r-k+i)}(x_i, \ldots, x_j) (k) \Omega_{(n-1)(j-i)}^{(r-k+i-1)}(x_i, \ldots, x_j)}.
\]
Conjecture 2. [Chepuri–L. 2020+] For $i < k \leq j - 1$, the following identity of $(k-1)\Omega$ and $(k)\Omega$ holds:

$$
\begin{align*}
\left[ \prod_{t=1}^{n-1} \sigma_{(n-1)(k-i)}^{(r-k+i+t)}(x_i, \ldots, x_k) \right]^{(k-1)}\Omega_{(n-1)(j-i)}^{(r-k+i)}(x_i, \ldots, x_j) \\
\sum_{s=0}^{n-1} \prod_{t=r+1}^{r+s} x_j^{(t+j-k)} \prod_{t=r+s+1}^{r+n-1} x_k^{(t+1)} \prod_{t=s+2}^{s+n-1} \sigma_{(n-1)(k-i)}^{(r-k+i+t)}(x_i, \ldots, x_k)
\end{align*}
$$

We proved this in the $n = 2$ case.
Future Directions

- Resolve the conjectures;
- Other permutations;
- Combinatorial interpretation of the $\Omega$ functions;
- Is there an easy way of interpreting the identities we are getting using a graphical calculus of cylindrical networks?
Acknowledgements

This research was conducted at the 2020 University of Minnesota Twin Cities REU, which was supported by NSF RTG grant DMS-1745638. We thank Pasha Pylyavskyy for proposing the problem and explaining his paper to us, and our TA Emily Tibor for her support, and her thoughtful and constructive feedback on this report and various presentations.

Lam, T. and Pylyavskyy, P. (2010). Intrinsic energy is a loop Schur function.