Periodic Infinite Friezes of Type $\Lambda_{p_1,\ldots,p_n}$ and Dissections on Annuli

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Abstract

We discussed the realization criterion of a reduced infinite frieze of type $\Lambda_{p_1,\ldots,p_n}$ and showed that periodic infinite friezes of type Λ_p that pass the realizability test are all realizable. For reduced infinite friezes of type $\Lambda_{p_1,\ldots,p_n}$ that pass the realizability test but do not have dissected annului correspondence, we introduce the quotient dissection of annuli as a geometric interpretation. We prove that for all friezes that pass the realizability test, their entries are in the form of sum of polygon path weights and that all realizable skeletal friezes are positive. We introduce a combinatorial interpretation of growth coefficients in skeletal friezes by a annulus weighting on polygon paths.

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1 Introduction

Finite frieze patterns of positive integers are first defined by Coxeter in [1]. This definition has been extend to the infinite cases by Baur, Parsons ,and Tschabold in [4], and entries in the frieze are generalized to algebraic integers by Holm and Jørgensen in [7]. In our report, we will use the definition of infinite frieze in the paper of BPS with slight modification:

Definition 1.0.1. An infinite frieze $\mathcal{F}$ is an array $(m_{i,j})_{i,j\in\mathbb{Z},j\geq i-2}$ of shifted infinite rows such that

- $m_{i,i-2} = 0$ and $m_{i,i-1} = 1$ for all $i \in \mathbb{Z}$;
- Every diamond in $\mathcal{F}$ of the form $\begin{matrix} & m_{i,j} & \\ m_{i-1,j} & & m_{i,j+1} \\ & m_{i+1,j+1} & \end{matrix}$ satisfies the unimodular rule: $m_{i-1,j}m_{i,j+1} - m_{i,j}m_{i+1,j+1} = 1$.

$$\begin{array}{ccccccccccccc}
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
\cdots & & 1 & & 1 & & 1 & & 1 & & & \cdots \\
\cdots & m_{-1,1} & & m_{0,2} & & m_{1,3} & & m_{2,4} & & m_{3,5} & & \cdots \\
\cdots & & m_{-1,2} & & m_{0,3} & & m_{1,4} & & m_{2,5} & & & \cdots \\
\cdots & m_{-2,2} & & m_{-1,3} & & m_{0,4} & & m_{1,5} & & m_{2,6} & & \cdots \\
\cdots & & m_{-2,3} & & m_{-1,4} & & m_{0,5} & & m_{1,6} & & & \cdots \\
 & & & \iddots & & m_{-1,5} & & \ddots & & & &
\end{array}$$

There is a well-know recurrence on frieze patterns that would be essential to proves in section 3. We will introduce it here.

$$m_{i,j} = m_{i,i+2}m_{i+1,j} - m_{i+2,j}$$

Definition 1.0.2. The first nontrivial row is called the *quiddity row*, indexed as row 1. Each frieze is uniquely defined by its quiddity row. If a quiddity row is periodic, then its corresponding frieze is periodic.

1.1 Infinite friezes of type $\Lambda_{p_1,p_2,\ldots,p_n}$

Holm and Jørgensen introduce a generalization of friezes of integers in [7].

Definition 1.1.1. Let $p \in \mathbb{Z} \geq 3$. A (finite) frieze pattern is of type λ_p if the quiddity row consists of (necessarily positive) integral multiples of

$$\lambda_p = 2\cos(\frac{\pi}{p})$$

We can generalize this definition to the infinite frieze case.

Definition 1.1.2. Let $p_1, \ldots, p_n \in \mathbb{Z} \geq 3$. An infinite frieze pattern is of type $\lambda_{p_1,\ldots,p_n}$ if each entry in its quiddity row is the sum of integral multiples of $\lambda_{p_1}, \ldots, \lambda_{p_n}$.
If every entry in the quiddity row of an infinite frieze $\mathcal{F}$ is the integral multiple of λ_p for a fixed $p \in \mathbb{Z} \geq 3$, then $\mathcal{F}$ is an infinite frieze of type Λ_p.

We can write each entry in the quiddity row in the form of

$$m_{i-1,i+1} = \sum_{p \in A_i} \lambda_p$$

where A_i is an indexing set that contains all integers p_s that sum up to $m_{i-1,i+1}$. Because multiples of a certain λ_{p_i} is allowed, we will record the repetition in A_i by indexing the repetitive integer p_i.

Lemma 1.1.1. If a sequence of numbers contains more than $p-2$ consecutive λ_p, then it cannot be a quiddity sequence of an infinite frieze pattern.

Proof. We can see $p-2$ consecutive λ_p as a period 1 frieze whose quiddity sequence is λ_p. By the results of Holm-Jørgensen, we know that entries in row k of this frieze are in the form of $U_k(\lambda_p)$. Moreover, the value $U_k(\lambda_p)$ reaches 0 when $k \geq p-1$. Based on the unimodular rule, $p-1$ consecutive λ_p will uniquely define an entry in row $p-1$ that only depends on these consecutive λ_p. Hence this frieze contains a term 0 and cannot be infinite. □

Corollary 1.1.1. If a sequence of numbers contains only λ_p for a certain p, then it is a quiddity sequence of a finite frieze pattern.

In this report, we will shorten the notion of *periodic infinite friezes of type* $\Lambda_{p_1,\ldots,p_n}$ as simply *infinite friezes*, unless else specified.

1.2 Dissected annulus

Conway and Coxeter have successfully established a bijection between finite friezes of positive integers with triangulations on polygons in [2][3]. This bijection is later extended to the case of infinite friezes of positive integers with triangulations on annuli and once punctured discs. Holm and Jørgensen proved a bijection between finite friezes of type Λ_p with p-angulations on polygons and an injection from dissection on polygons to finite friezes of type $\Lambda_{p_1,..,p_n}$. Following this progression of understanding on frieze patterns, this project explored the correspondence between dissection on annuli and once punctured disks and infinite friezes of type.

Definition 1.2.1. We will let $A_{n,m}$ denote an annulus with n vertices on the outer boundary and m vertices on the inner boundary, and a once punctured disc with n vertices on the boundary corresponds to $A_{n,0}$. Because in most cases only the outer boundary vertices are essential, we will denote an annulus with n outer boundary vertices as A_n for short, and a vertex would mean an outer boundary vertex, denoted as v_i, unless else specified. On such an annulus, the outer vertex would be indexed clockwise, modulo n. For instance, $v_{n+i} = v_i$.

Let $\mathcal{D}$ be a polygon dissection on the annulus A_n, for each vertex v_i, we will let Poly(v_i) denote the set of polygons $\mathcal{P}$ in $\mathcal{D}$ that incident to vertex v_i. For each subgon $\mathcal{P}$ in $\mathcal{D}$, we will use $|\mathcal{P}|$ denote the number of edges of this subgon.

It is conventional to consider an annulus in its universal cover, which is an infinite stripe. The idea of mapping a triangulation on an annulus to a periodic triangulation on its universal cover is explored in [5]. Using similar analogy, we can map a dissection on an annulus to a periodic dissection on an infinite stripe, where the bottom line represents the outer boundary and the top line represents the inner boundary. While v_i and v_{i+n} refer to the same vertex on an annulus, they are distinct on the infinite strip. Let ρ be the covering map that maps the infinite stripe dissection down onto the dissected disk, then $\rho(v_i) = \rho(v_{i+kn}) = v_i$ on the annulus.

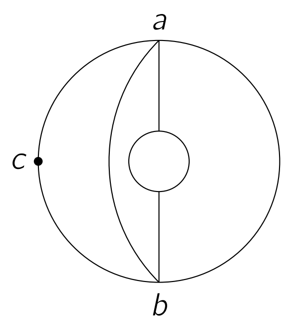
Lemma 1.2.1. Let $\overline{\mathcal{D}}$ be a polygon dissection on an infinite strip, then for each lower boundary edge (v_i, v_{i+1}), there exists exactly one subgon in $\overline{\mathcal{D}}$ that contains (v_i, v_{i+1}). This would be the only subgon incident to both v_i and v_{i+1}

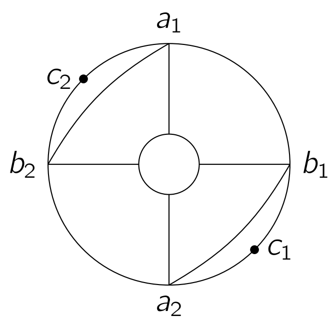
Definition 1.2.2. Let $\mathcal{D}$ be a dissection on an annulus $A_{n,m}$, we can define

$$\textit{the } k^{th} \textit{ power of } \mathcal{D} \textit{ on } A_{kn,km}$$

by dissecting an annulus $A_{kn,km}$ by repeating $\mathcal{D}$ exactly k times.
For any $k \in \mathbb{Z} > 1$, the k^{th} power of $\mathcal{D}$ has the exact same infinite strip representation as $\mathcal{D}$.

Example 1.2.1. Square of a dissection example.





Definition 1.2.3. Let $\mathcal{D}$ be a polygon dissection on the annulus A_n. An arc connecting an outer vertex and an inner vertex is called a *bridging arc*. An arc connecting two outer vertices is called a *peripheral arc*. We do not allow arcs to connect two inner vertices in the dissection.

Baur et al. defined a skeletal triangulation which only consists of bridging arcs in [5]. We will extend this definition to dissections.

Definition 1.2.4. A dissection is said to be *skeletal* if it contains only bridge arcs. In a skeletal dissection, a subgon that contains at least one outer edge (v_i, v_{i+1}) is called an *outer subgon*. A subgon that contains no outer edge would be called as an *inner subgon*. Note that an inner subgon in skeletal dissections always incident to exactly 1 outer vertex. In a dissection that is not skeletal, a subgon $\mathcal{P}$ is called as an *ear* or an *peripheral subgon* if it consists of $|\mathcal{P}| - 1$ outer boundary edges and exactly one peripheral arc.

Lemma 1.2.2. If $\mathcal{D}$ is a skeletal dissection,then it does not contain any ears.

To establish a correspondence, following the previous study, if an outer boundary vertex in a dissected annulus is adjacent to polygons of size $p_1, \ldots, p_n$, we associate to it the weighted count $\sum_{i=1}^{n} \lambda_{p_i}$ in the quiddity row. Baur et al. used both the inner and outer boundaries of an annulus for this correspondence in [], but this project is only focusing on the outer boundary. Hence, we exclude all peripheral arcs that connect inner vertices because they are trivial for our correspondence. In section 2, we will discuss the realization criterion of reduced infinite friezes and will introduce a new type of dissection for some friezes that do not have dissected annuli correspondence.

Broline, Crowe and Isaacs found a correspondence between all entries in finite friezes of positive integers and triangulations on polygon by using matching numbers. This correspondenc is then extended to the infinite friezes of positive integers cases by Baur, Parsons, and Tschabold in [4]. Bessenrodt introduced polynomially weighted walks around such dissected polygons and their correspondence with entries in finite frieze of type $\Lambda_{p_1,\ldots,p_n}$ in [9]. We generalized this notion of "matching numbers" to weights on polygon paths in dissected annuli. A correspondence between all entries in infinite friezes and this weights on path is proved in section 3.

Definition 1.2.5. Let $\mathcal{D}$ be a dissection of a punctured disc or an annulus with n vertices, then a *path from vertex i to vertex j*, $w_{i,j}$ would be a sequence:

$$w_{i,j} = (\mathcal{P}_i, \mathcal{P}_{i+1}, \ldots, \mathcal{P}_{j-1}, \mathcal{P}_j)$$

of subgons in the corresponding dissection on the infinite strip, such that $\mathcal{P}_i$ incident to vertex v_i on the infinite strip.

- We will use $w[a:b] = (\mathcal{P}_a, \ldots, \mathcal{P}_b)$ to denote a subpath of w, where $i \leq a \leq b \leq j$.
- We will use $\mathbf{P}_{i,j}$ to denote the set of all paths from v_i to v_j.

Definition 1.2.6. For each path $w_{i,j} = (\mathcal{P}_i, \mathcal{P}_{i+1}, \ldots, \mathcal{P}_{j-1}, \mathcal{P}_j)$, we will define its *path weight* $wt(w_{i,j})$ as the following:

- Weight of a 0 length path would have weight 1;
- $wt(w_{i,i}) = \lambda_{|\mathcal{P}_i|}$;
- If $\mathcal{P}_i = \mathcal{P}_{i+1}$, then $wt(w_{i,j}) = wt((\mathcal{P}_i))wt((\mathcal{P}_{i+1}, \ldots, \mathcal{P}_j)) - wt((\mathcal{P}_{i+2}, \ldots, \mathcal{P}_j))$;
- If $\mathcal{P}_i \neq \mathcal{P}_{i+1}$, then $wt(w_{i,j}) = wt((\mathcal{P}_i))wt((\mathcal{P}_{i+1}, \ldots, \mathcal{P}_j))$

We will use $U_k(x)$ to denote the Chebyshev polynomials of the second kind with the following properties:

- $U_0(x) = 1$;
- $U_1(x) = x$;

- $U_k(x) = U_1(x)U_{k-1}(x) - U_{k-2}(x)$.

Lemma 1.2.3. A path $w_{i,j} = ((\mathcal{P}_i, ..., \mathcal{P}_j))$ can be partitioned into a sequence of subpath $(\mathcal{P}_i, ..., \mathcal{P}_{a_1})$, $(\mathcal{P}_{a_1+1}, ..., \mathcal{P}_{a_2})$, $(\mathcal{P}_{a_2+1}, ..., \mathcal{P}_{a_3})$,..., $(\mathcal{P}_{a_{h-1}+1}, ..., \mathcal{P}_{a_h})$, $(\mathcal{P}_{a_h+1}, ..., \mathcal{P}_j)$ such that:

- Let $a_0 = i - 1$, $a_{h+1} = j$. For each subpath $(\mathcal{P}_{a_s+1}, ..., \mathcal{P}_{a_{s+1}})$, we have $\mathcal{P}_{a_s+1} = \mathcal{P}_{a_s+2} = ... = \mathcal{P}_{a_{s+1}}$

- For 2 neighboring subpaths $(\mathcal{P}_{a_{s-1}+1}, ..., \mathcal{P}_{a_s})$ and $(\mathcal{P}_{a_s+1}, ..., \mathcal{P}_{a_{s+1}})$, we have $\mathcal{P}_{a_s} \neq \mathcal{P}_{a_s+1}$

And that

$$w_{i,j} = \prod_{t=0}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|})$$

where $U_k(x)$ is the Chebyshev polynomial.

Proof. The partition essentially divides the path into subpaths consist of consecutive identical subgons to the maximum. We will prove by induction on length l of paths.
By definition of $U_k(x)$, we know that $U_0(x) = 1$ and $U_1(x) = x$. Thus the argument holds true for path of length $= 0$ and $l = 1$.
Suppose that the argument holds true for all paths with length less or equal to l and let $w = (\mathcal{P}_i, ..., \mathcal{P}_{i+l})$ be any path with length $l + 1$.

- If $\mathcal{P}_i \neq \mathcal{P}_{i+1}$, then $a_1 = i$, the first subpath in the partition is $(\mathcal{P}_i)$

$$\begin{aligned} wt((\mathcal{P}_i, ..., \mathcal{P}_{i+l})) &= \lambda_{|\mathcal{P}_i|} wt((\mathcal{P}_{i+1}, ..., \mathcal{P}_{i+l})) \\ &= U_1(|\mathcal{P}_i|) \prod_{t=1}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|}) \\ &= \prod_{t=0}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|}) \end{aligned}$$

- If $\mathcal{P}_i = \mathcal{P}_{i+1}$, then $a_1 \geq i + 1$, denote the length of the first subpath $(\mathcal{P}_i, ..., \mathcal{P}_{a_1})$ as l_0.

i.e. there are l_0 consecutive $\mathcal{P}_i$ at the beginning of w.

$$\begin{aligned}
wt((\mathcal{P}_i, ..., \mathcal{P}_{i+l})) =& \lambda_{|\mathcal{P}_i|} wt((\mathcal{P}_{i+1}, ..., \mathcal{P}_{i+l})) - wt((\mathcal{P}_{i+2}, ..., \mathcal{P}_{i+l})) \\
=& U_1(\lambda_{|\mathcal{P}_i|}) \left(U_{l_0-1}(\lambda_{|\mathcal{P}_i|}) \prod_{t=1}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|}) \right) \\
&- \left(U_{l_0-2}(\lambda_{|\mathcal{P}_i|}) \prod_{t=1}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|}) \right) \\
=& (U_1(\lambda_{|\mathcal{P}_i|}) U_{l_0-1}(\lambda_{|\mathcal{P}_i|}) - U_{l_0-2}(\lambda_{|\mathcal{P}_i|})) \prod_{t=1}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|}) \\
=& U_{l_0}(\lambda_{|\mathcal{P}_i|}) \prod_{t=1}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|}) \\
=& \prod_{t=0}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|})
\end{aligned}$$

□

2 Periodic infinite friezes of type $\Lambda_{p_i, p_{i+1}, .., p_n}$ and dissected annuli realizations

Definition 2.0.1. An infinite frieze pattern of period n is called *realizable* if there exists a dissection $\mathcal{D}$ on an annulus A_n with n vertices if for each entry $m_{i-1,i+1}$ in the quiddity sequence,

$$m_{i-1,i+1} = \sum_{w \in \mathrm{P}_{i,i}} wt(w) = \sum_{p \in \mathrm{Poly}(v_i)} \lambda_{|\mathcal{P}|}$$

A frieze pattern that has such a geometric interpretation is called *realizable*.

Remark 2.0.1. If we read off entries from the periodic dissection on the infinite stripe developed based on the realization of an infinite frieze pattern by the same formula,

$$m_{i-1,i+1} = \sum_{w \in \mathrm{P}_{i,i}} wt(w) = \sum_{p \in \mathrm{Poly}(v_i)} \lambda_{|\mathcal{P}|}$$

we would obtain exactly the quiddity row of this infinite frieze.

Frieze patterns are uniquely defined by their quiddity rows, and we can tell from certain quiddity sequences that the friezes developed from these quiddity sequences are not realizable.

Proposition 2.0.1 (Realizability test)**.** Entries in the quiddity row can be written in the form of $m_{i-1,i+1} = \sum_{p\in A_i} \lambda_p$. Let $m_{i-1,i+1}$, $m_{i,i+2}$ be a pair of neighboring entries in the quiddity row. If $A_i \cap A_{i+1} = \emptyset$, then this frieze pattern is not realizable.

Proof. Let $\mathcal{F}$ be a frieze pattern that contains two neighboring quiddity entries

$$m_{i-1,i+1} = \sum_{p\in A_i} \lambda_p, m_{i,i+2} = \sum_{p\in A_{i+1}} \lambda_p$$

such that $A_i \cap A_{i+1} = \emptyset$.
Assume for the purpose of contradiction that $\mathcal{F}$ is realizable, then its quiddity entries can be written in the form of $m_{i-1,i+1} = \sum_{\mathcal{P}\in \mathrm{Poly}(i)} \lambda_{|\mathcal{P}|}$, and we can consequentially replace A_i by Poly(i) as the indexing set.
Let the dissection $\mathcal{D}$ on an annulus A_n is one of its corresponding dissection. Then there must exists a subgon in $\mathcal{D}$ that contains the outer boundary edge (v_i, v_{i+1}). Denote this subgon as $\hat{\mathcal{P}}$.
Therefore we know that $\hat{\mathcal{P}} \in \mathrm{Poly}(v_i) \cap \mathrm{Poly}(v_{i+1})$, so $\mathrm{Poly}(v_i) \cap \mathrm{Poly}(v_{i+1}) \neq \emptyset$. We reached a contradiction. □

We would call the process of checking all adjacent pairs in a quiddity sequence whether the pair would indicate the unrealizability of the frieze pattern based on Proposition as the *realizability test*.

Conjecture 2.0.1. A frieze that fails the realizability test would contain negative entries.

Corollary 2.0.1. Let $\mathcal{F}$ be a frieze pattern with a quiddity sequence

$$q = (m_{0,2}, m_{1,3}, \ldots, m_{n-1,n+1})$$

that contains consecutive λ_p $m_{i-1,i+1} = m_{i,i+2} = \ldots = m_{j-1,j+1} = \lambda_p$ for some i, j but $m_{i-2,i} \neq \lambda_p$, $m_{j,j+2} \neq \lambda_p$. Write the neighboring entries as

$$m_{i-2,i} = \sum_{k\in A_{i-1}} \lambda_k, m_{j,j+2} = \sum_{n\in A_{j+1}} \lambda_n$$

If $p \notin A_{i-1} \cap A_{j+1}$, then $\mathcal{F}$ is not realizable.

An arbitrary infinite frieze that passes the realizability test is not guaranteed to be realizable. However, in section 2.1 we would show that every reduced frieze would have *some* geometric interpretation.
We can perform the following algorithm to obtain a reduced frieze from an arbitrary infinite frieze.

Definition 2.0.2. Let $\mathcal{F}$ be a frieze pattern that passed the realizability test and let $q = (m_{0,2}, m_{1,3}, ..., m_{n-1,n+1})$ be its quiddity sequence.
A *cut* can be performed on q if there exists some $i \in [n]$ and some integer p such that $m_{i,i+2} = m_{i+1,i+3} = \ldots = m_{i+p-3,i+p-1} = \lambda_p$.
A cut on q would replace the subsequence

$$(m_{i-1,i+1}, m_{i,i+2}, ..., m_{i+p-3,i+p-1}, m_{i+p-2,i+p})$$

by

$$(m_{i-1,i+1} - \lambda_p, m_{i+p-2,i+p} - \lambda_p)$$

Remark 2.0.2. Suppose that $\mathcal{F}$ is realizable, and that its quiddity row contains $q-2$ consecutive λ_q, $m_{i,i+2} = m_{i+1,i+3} = ... = m_{i+p-3,i+p-1} = \lambda_p$, then the outer boundary edges $(i, i+1)$, $(i+1, i+2)$,..., $(i+p-2, i+p-1)$ are contained in one p-subgon, meaning that this p-subgon is an ear identified by the peripheral arc $(i, i+p-1)$. A cut on the quiddity row would cut this ear off from the dissection. leaving a new dissection on annulus that correspond to the frieze pattern generated by the quiddity row after the cut. Recursively cutting the quiddity row would eventually leave us with a *skeletal frieze*.

Definition 2.0.3. Let $\mathcal{F}$ be an infinite frieze that may or may not contain terms in the form of λ_p in its quiddity row. If terms in the form of λ_p for a fixed p always occur less than $p-2$ times in the quiddity row of $\mathcal{F}$, we say that $\mathcal{F}$ is a skeletal frieze.

Lemma 2.0.1. A skeletal dissection of an annulus would generate a realizable skeletal frieze pattern and vice versa.

This operation of a *cut* is first introduced by Holm and Jørgensen in [7] to delete subgons from dissections on polygons. In [5], Baur et al. defined the cut operation on dissected annulus to delete subgons and peripheral arcs. The cut operation is the only reductive operation they defined, which is sufficient for friezes of integers to get rid of 1 in the quiddity row. For infinite friezes of type $\Lambda_{p_1,...,p_n}$, however, we would need another operation to completely reduce an infinite frieze.

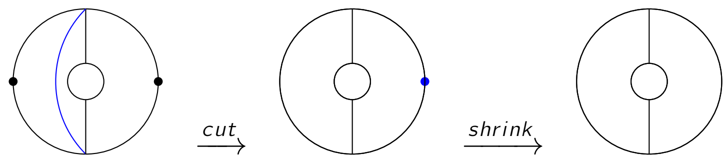
Definition 2.0.4. Let $\mathcal{F}$ be a frieze pattern that passed the realizability test. Let $q = (m_{0,2}, m_{1,3}, ..., m_{n-1,n+1})$ be its quiddity sequence.
A *shrink* can be performed on q if there exists some $i \in [n]$ and some integer p such that $m_{i,i+2} = m_{i+1,i+3} = ... = m_{i+k-1,i+k+1} = \lambda_p$, where $k < p-2$.
A shrink on q would replace the subsequence

$$(m_{i-1,i+1}, m_{i,i+2}, ...m_{i+k-1,i+k+1}, m_{i+k,i+k+2})$$

by

$$(m_{i-1,i+1} - \lambda_p + \lambda_{p-k}, m_{i+k,i+k+2} - \lambda_p + \lambda_{p-k})$$

Remark 2.0.3. Suppose that $\mathcal{F}$ is realizable, and that its quiddity row contains k consecutive λ_p, $m_{i,i+2} = m_{i+1,i+3} = \ldots = m_{i+k-1,i+k+1} = \lambda_p$, then the outer boundary edges $(i, i+1), (v_{i+1}, v_{i+2}), \ldots, (v_{i+k}, v_{i+k+1})$ are contained in one p-subgon that has exactly $k+2$ outer vertices and $p-k-2$ inner vertices. A shrink on this quiddity row would delete the vertices $v_{i+1}, v_{i+2}, \ldots, v_{i+k}$ from the outer boundary and hence from this p-subgon, leaving a shrunk subgon consisting of outer vertices v_i, v_{i+k+1} and $p-k-2$ inner vertices. Because $k < p-2$, the shrunk subgon would still be a polygon.



Lemma 2.0.2. If an infinite frieze $\mathcal{F}$ is realizable, then its reduced frieze is realizable.

Remark 2.0.4. The converse of this statement is in fact not true. For example, the quiddity sequence $q = (\lambda_4 + 2\lambda_5, \lambda_5, \lambda_5, \lambda_4 + \lambda_5, 2\lambda_5)$ is unrealizable, but it can be reduced to a realizable quiddity sequence $(\lambda_3 + \lambda_4 + \lambda_5, \lambda_3 + \lambda_4, \lambda_3 + \lambda_5)$.

Definition 2.0.5. A *reduction (at consecutive λ_p)* of a quiddity sequence

$$q = (m_{0,2}, m_{1,3}, \ldots, m_{n-1,n+1})$$

of a frieze $\mathcal{F} = \mathcal{F}^0$ is obtained by recursively performing the following algorithm:

1. If $\mathcal{F}^n$ failed the realizability test, then $\mathcal{F}^n$ cannot be reduced and thus $\mathcal{F}^0$ is not realizable.

2. If $\mathcal{F}^n$ passed the realizability test, we would delete the entries $m_{i-1,i+1}$ that of the form λ_p through cutting and shrinking, obtaining a new frieze $\mathcal{F}^{n+1}$.

3. Repeat step 1 and 2 until no entries in the quiddity row is of the form λ_p. We would call such quiddity sequence as a *reduced quiddity sequence* and the frieze a *reduced frieze*, denoted as $\mathcal{F}'$.

Definition 2.0.6. Let $\mathcal{D}$ be a dissection on annulus A_n. If for all outer vertices v_i, there are at least 2 subgons in $\mathcal{D}$ that incident to v_i, then we say $\mathcal{D}$ is a *reduced dissection.*

2.1 Realizations of reduced infinite friezes

Let $\mathcal{F}$ be a reduced periodic infinite frieze of type $\Lambda_{p_i, p_{i+1}, \ldots, p_n}$. Again, we can perform the realizability test on $\mathcal{F}$. If $\mathcal{F}$ failes, then we know that it is not realizable.

Before we discussed the realizability of the reduced frieze, we would first discuss some properties of the reduced frieze and its corresponding annulus dissection if the reduced frieze is realizable.

Lemma 2.1.1. If a reduced frieze $\mathcal{F}$ is realizable, then its corresponding annulus dissection $\mathcal{D}$ is skeletal.

Proof. Suppose not, then $\mathcal{D}$ would contain as least one ear subgon p, bounded by the peripheral arc (v_i, v_j), and $j \geq i+2$. Therefore, p is the only subgon in $\mathcal{D}$ that incident to v_{i+1}, so $m_{i,i+2} = \lambda_{|\mathcal{P}|}$. We reached a contradiction. □

Remark 2.1.1. Note that although reduced friezes are all skeletal, not all skeletal friezes are reduced.

Lemma 2.1.2. If a reduced frieze $\mathcal{F}$ is realizable and $\mathcal{D}$ is its corresponding dissection, then each outer subgon in $\mathcal{D}$ contains exactly one outer edge.

Proof. Suppose not, let $\mathcal{P}$ be a subgon in $\mathcal{D}$ that contains more than 1 outer edge. Let (v_i, v_{i+1}) be one of the outer edges contained in $\mathcal{P}$. Then there exists the outer edge (v_j, v_{j+1}) that is contained in $\mathcal{P}$ and that for all edges (v_k, v_{k+1}) where $i < k < j$, (v_k, v_{k+1}) is not in $\mathcal{P}$.
If $i + 1 = j$, then because $\mathcal{D}$ is skeletal and any bridge arc incident to v_{i+1} would separate (v_i, v_{i+1}), (v_{i+1}, v_{i+2}) into 2 different subgons, there is no arc incident to v_{i+1}. $\mathcal{P}$ is the only subgon incident to v_{i+1}, so $m_{i+1-1,i+1+1} = \lambda_{|\mathcal{P}|}$. We reached a contradiction.
If $i + 1 < j$, by previous argument we know that edges (v_{i-1}, v_i) and (v_{j+1}, v_{j+2}) cannot be in p. Therefore, there would be 4 bridge arcs $e_{i-1}, e_i, e_j, e_{j+1}$, incident to $v_{i-1}, v_i, v_j, v_{j+1}$ respectively. However, in this case, (v_i, v_{i+1}) and (v_j, v_{j+1}) would belong to two different subgons. Hence we reached a contradiction. □

Corollary 2.1.1. If a reduced frieze $\mathcal{F}$ is realizable and that $\mathcal{D}$ is its corresponding dissection, then different outer boundary edges in $\mathcal{D}$ are contained in different outer subgons.

Proposition 2.1.1. Let $\mathcal{F}$ be a reduced frieze, and $q = (m_{0,2}, m_{1,3}, \ldots, m_{n-1,n+1})$ be its quiddity sequence. $\mathcal{F}$ is realizable if and only if there exists a sequence of n numbers $p_{1,2}, p_{2,3}, \ldots, p_{n,n+1}$ such that $p_{i,i+1} \in A_i \cap A_{i+1}$ for all $i \in [n]$, such that if $p_{i-1,i} = p_{i,i+1}$ numerically, there are at least two copies of the number $p_{i-1,i}$ in A_i.

Proof. We will first prove the only if direction. Suppose $\mathcal{F}$ is realizable and let $\mathcal{D}$ denotes a corresponding dissection on annulus, then by Corollary 2.1.1, each outer edge (v_i, v_{i+1}) is contained in a distinct outer subgon $\mathcal{P}_{i,i+1}$. Thus, $|\mathcal{P}_{1,2}|, |\mathcal{P}_{2,3}|, \ldots, |\mathcal{P}_{n,n+1}|$ can be the sequence we want.
For each tuple (v_{i-1}, v_i, v_{i+1}), there are two different subgons $\mathcal{P}_{i-1,i}, \mathcal{P}_{i,i+1} \in \mathcal{D}$ such that

$\mathcal{P}_{i-1,i}$ contains the edge (v_{i-1}, v_i) and $\mathcal{P}_{i,i+1}$ contains the edge (v_i, v_{i+1}).
Therefore, $|\mathcal{P}_{i-1,i}| \in A_{i-1} \cap A_i$, $|\mathcal{P}_{i,i+1}| \in A_i \cap A_{i+1}$, and $|\mathcal{P}_{i-1,i}|, |\mathcal{P}_{i,i+1}| \in A_i$. Recall that identical integers can have repetitions in A_i, so that even if $|\mathcal{P}_{i-1,i}| = |\mathcal{P}_{i,i+1}|$ numerically, they would have 2 different representations in A_i.
For the if direction, we would construct a corresponding dissection for an arbitrary reduced frieze $\mathcal{F}$ of period n that satisfies the condition in the proposition. We will first start from determining the shape of each of the n outer subgons.
Let $p_{1,2}, p_{2,3}, \ldots, p_{n,n+1}$ be a sequence of numbers that satisfies the requirement in the proposition. Then for the outer edge (v_i, v_{i+1}), there are a pair of bridge arcs (v_i, u) and (v_{i+1}, u') such that:

- There is no other bridge arc between (v_i, u) and (v_{i+1}, u') counting clockwise ;
- If $u \neq u'$, there are exactly $p_{i,i+1} - 4$ inner vertices between u and u'; if $u = u'$, there are exactly $p_{i,i+1} - 4$ inner vertices between u and u'. These vertices, along with u, u', v_i, v_{i+1}, would construct a $p_{i,i+1}$-subgon.

For each quiddity entry $m_{i-1,i+1}$, we know that $p_{i-1,i}, p_{i,i+1} \in A_i$, and that

$$m_{i-1,i+1} = \lambda_{p_{i-1,i}} + \lambda_{p_{i-1,i}} + \sum_{q \in A_i \setminus \left\{ p_{i-1,i}, p_{i,i+1} \right\}} \lambda_q$$

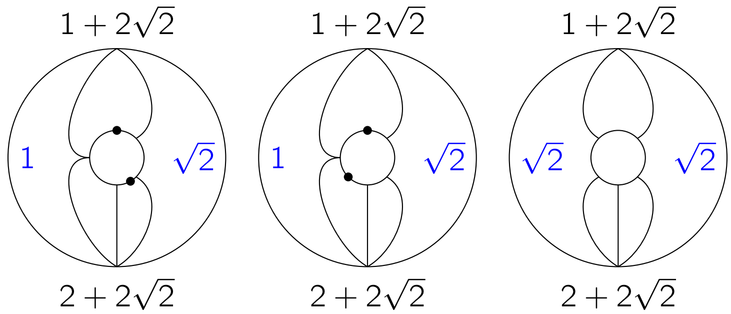
We have just constructed the only two outer subgons that incident to v_i that contribute to the terms $\lambda_{p_{i-1,i}} + \lambda_{p_{i-1,i}}$, and now we will use inner subgons that incident to v_i for the terms $\sum_{q \in A_i \setminus \left\{ p_{i-1,i}, p_{i,i+1} \right\}} \lambda_q$. There would be exactly $|A_i| - 2$ inner subgons incident to v_i. All of these inner subgons are contained between the two outer subgons that incident to v_i. The number of edges for these $|A_i| - 2$ inner subgons would correspond to numbers in $A_i \setminus \left\{ p_{i-1,i}, p_{i,i+1} \right\}$ respectively.
If $|A_i| - 2 = 0$, then the two outer subgons that incident to (v_i) would share one bridge arc that incident to v_i.
By this construction, for each quiddity entry $m_{i-1,i+1}$, we would have:

$$\begin{aligned}
\sum_{\substack{p \text{ is a subgon} \\ \text{incident to } v_i}} \lambda_p &= \sum_{\substack{p \text{ is a subgon} \\ \text{incident to } v_i}} \lambda_p + \sum_{\substack{p \text{ is a subgon} \\ \text{incident to } v_i}} \lambda_p \\
&= \lambda_{p_{i-1,i}} + \lambda_{p_{i,i+1}} + \sum_{q \in A_i - \left\{ p_{i-1,i}, p_{i,i+1} \right\}} \lambda_q \\
&= \sum_{q \in A_i} \lambda_q \\
&= m_{i-1,i+1}
\end{aligned}$$

□

Remark 2.1.2. Because there might be multiple sequence of n numbers $p_{1,2}, p_{2,3}, \ldots, p_{n,n+1}$ that satisfy the conditions, there might be multiple realization of a certain quiddity sequence.

Example 2.1.1. Dissection of annulus corresponding to $q = (1+2\sqrt{2}, 2+2\sqrt{2})$



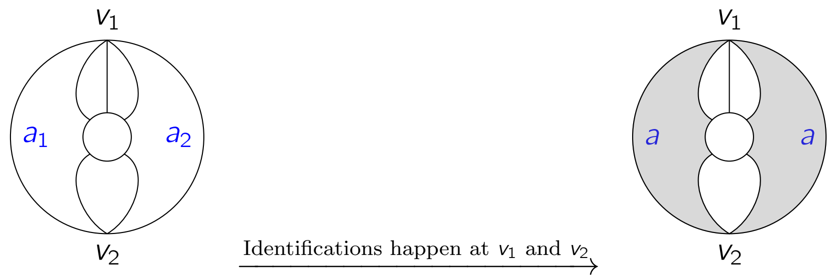
Corollary 2.1.2. A reduced frieze of Type Λ_p is always realizable.

Proof. Let $\mathcal{F}$ be a reduced frieze of Type Λ_p. We can always choose a default sequence of $(\lambda_p, \lambda_p m, \ldots, \lambda_p)$. Because $\mathcal{F}$ is reduced, all of its quiddity entries would be in the form of $k\lambda_p$ where $p \geq 2$. Thus, we always have at least 2 copies of p in the indexing sets. □

Notice that there are reduced friezes that pass the realizability test but do not satisfy the requirements in Proposition 2.1.1. For instance, the quiddity sequence $(\lambda_3+\lambda_5, \lambda_3+\lambda_4, \lambda_3+\lambda_4, \lambda_4+\lambda_5)$. We do not have a normal dissected annulus interpretation of these reduced friezes, but we can construct another type of geometric interpretation for them.

Definition 2.1.1. Let $\mathcal{D}$ be a skeletal dissection on an annulus A_n whose corresponding frieze is a reduced frieze. If for some outer vertex v_i, its incident outer edges v_{i-1}, v_i and (v_i, v_{i+1}) are contained in two different p-subgons for an integer p, we can identify these two p-subgons as one. The geometric object obtained by such identification is called a *quotient dissection on annulus.*
We allow multiple p-subgons with the same p to be identified as one as long as they are consecutive outer subgons.
If the two outer subgons incident to the vertex i are identified as one, we say that an *identification happens at* v_i.

Example 2.1.2. Quotient dissection



Definition 2.1.2. We say that a frieze pattern $\mathcal{F}$ has a quotient dissection realization if there exists some quotient dissection $\mathcal{D}$ on an annulus such that for every entry in the quiddity sequence, we have:

$$m_{i-1,i+1} = \sum_{w \in \mathrm{P}_{i,i}} wt(w) = \sum_{p \in \mathrm{Poly}(v_i)} \lambda_{|\mathcal{P}|}$$

where p are subgons in $\mathcal{D}$, v_i are outer vertices on the quotient annulus, and w are polygon path in $\mathcal{D}$.

We want to construct a periodic quotient dissection $\overline{\mathcal{D}}$ on the infinite strip that persists the same property described in Remark 2.0.1. We would start from the periodic dissection $\overline{\mathcal{D}_0}$ developed based on the normal dissection $\mathcal{D}_0$ from which our quotient dissection $\mathcal{D}$ is constructed. If p, q are two outer subgons in $\mathcal{D}_0$ that incident to v_i and are identified as one in $\mathcal{D}$, we can simply identify the periodic pair p_k, q_k that incident to v_{i+kn} in $\overline{\mathcal{D}_0}$ as one subgon in order to obtain $\overline{\mathcal{D}}$. It is worth noticing, however, that in the case where all n outer subgons in $\mathcal{D}_0$ are all p-gons for some p and are identified as one in $\mathcal{D}$, all copies of this outer subgon in $\overline{\mathcal{D}_0}$ would be identified as one in $\overline{\mathcal{D}}$.

Proposition 2.1.2. Let $\mathcal{F}$ be a reduced frieze. If $\mathcal{F}$ passes the realizability test, then $\mathcal{F}$ is realizable either by a normal dissection on annulus as discussed in Proposition 2.1.1 or by a quotient dissection on annulus.

Proof. Because $\mathcal{F}$ passed the realizability test, there exists a sequence of n numbers $p_{1,2}, p_{2,3}, \ldots, p_{n,n+1}$ such that $p_{i,i+1} \in A_i \cap A_{i+1}$ for all $i \in [n]$. If $\mathcal{F}$ is not realizable by a normal dissection, then there exist some i such that $p_{i-1,i} = p_{i,i+1}$ numerically but only one copy of the number $p_{i-1,i}$ is in A_i.

Fix the sequence $p_{1,2}, p_{2,3}, \ldots, p_{n,n+1}$. We would construct a new quiddity sequence q' by adding $\lambda_{p_{i-1,i}}$ to $m_{i-1,i+1}$ for each i that do not satisfy the condition in Proposition 2.1.1. Therefore, there would exist exactly 2 copies of $p_{i-1,i} = p_{i,i+1}$ in A_i'. Consequentially, for the same sequence $p_{1,2}, p_{2,3}, \ldots, p_{n,n+1}$, now we would have that if there exist some i such that $p_{i-1,i} = p_{i,i+1}$ numerically, there would be at least 2 copies of $p_{i-1,i} \in A_i$. Thus, q' is

realizable. Let the corresponding normal dissection be $\mathcal{D}_0$.
For each i such that $p_{i-1,i} = p_{i,i+1}$ numerically but only one copy of the number $p_{i-1,i}$ is in A_i, we have

$$m_{i-1,i+1} = \lambda_{p_{i-1,i}} + \sum_{q \in A_i, q \neq p_{i-1,i}} \lambda_q$$

and

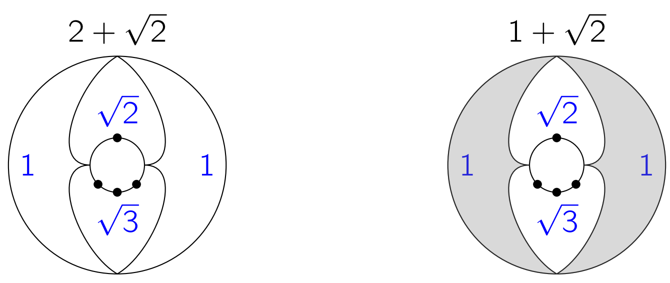
$$m'_{i-1,i+1} = 2\lambda_{p_{i-1,i}} + \sum_{q \in A'_i, q \neq p_{i-1,i}} \lambda_q = 2\lambda_{p_{i-1,i}} + \sum_{q \in A_i, q \neq p_{i-1,i}} \lambda_q$$

In $\mathcal{D}_0$, the subgons incident to v_i that contribute to λ_q, where $q \in A'_i$, $q \neq p_{i-1,i}$ are all inner subgons by construction. The only 2 $p_{i-1,i}$-subgons that incident to v_i are the two outer subgons. We would then identify these two subgons as one. We continue to do the identification for all i where $\lambda_{p_{i-1,i}}$ has been added to $m_{i-1,i+1}$, and call the resulting quotient dissection as $\mathcal{D}$.
We can read off a quiddity sequence $\overline{q}$ from $\mathcal{D}$ by the same rule as the normal dissection cases, which is:

$$\overline{m_{i-1,i+1}} = \sum_{p \in \mathrm{Poly}(v_i)} \lambda_{|\mathcal{P}|}$$

For all v_j such that an identification does not happen at v_j, $\overline{m_{j-1,j+1}} = m'_{j-1,j+1} = m_{j-1,j+1}$; for all v_j where an identification happened, $\overline{m_{j-1,j+1}} = m'_{j-1,j+1} - \lambda_{q_{j-1,j}} = m_{j-1,j+1}$. □

Example 2.1.3. We will realize a period 2 unrealizable quiddity sequence $q = (1+\sqrt{3},\ 1+\sqrt{2})$ by first realizing $q' = (\ 2 + \sqrt{3},\ 2 + \sqrt{2}\ )$.



2.2 Realizations of periodic infinite friezes of type Λ_p

Proposition 2.2.1. Every periodic infinite frieze of type Λ_p is realizable by a normal dissection on annulus.

Proof. We can simplify our process of construction in the case of frieze of Λ_p. In the reduction algorithm, we only need to cut off all of the peripheral arcs. From a frieze of Λ_p that no more than $p-3$ consecutive terms λ_p are contained in its quiddity row, we can directly construct its corresponding normal p-angulation by the following rule:

- If the quiddity row contains k consecutive terms λ_p, $k < p-2$,

  $$(m_{i,i+2}...m_{i+k-1,i+k+1})$$

  then the corresponding outer subgon would contain the vertices $v_i, v_{i+1}, ..., v_{i+k}$ and p-k-2 vertices on the inner circle.

- If two neighboring entries in the quiddity row are in the form of

  $$m_{k-1,k+1} = a\lambda_p, \ m_{k,k+2} = b\lambda_p$$

  where $a > 1$ and $b > 1$, then the outer subgon that contains the edge (v_k, v_{k+1}) would contain $p-2$ inner circle vertices.

- After we determined the outer subgons, we can keep adding inner subgons at each outer vertex so that the numbers match up with the quiddity row.

□

Conjecture 2.2.1. The realization of periodic infinite frieze of type Λ_p is essentially unique up to rotating the inner boundary and the location of the vertices.

3 Combinatorial interpretation of nontrivial entries

In this section, we would discuss the nontrivial entries in a realizable frieze that are not in the quiddity row. Let $\mathcal{F}$ be a realizable infinite frieze and let $\mathcal{D}$ be one of its corresponding dissection on annulus. We know that $\mathcal{F}$ is uniquely determined by its quiddity row q, and because information in the quiddity row is recorded in $\mathcal{D}$, it is reasonable to assume that from the dissection $\mathcal{D}$, we can read off all entries in $\mathcal{F}$.

Theorem 3.0.1. Every nontrivial entry in a realizable infinite frieze pattern of period n that correspond to a quotient dissection $\mathcal{D}$ satisfies that

$$m_{i-1,j+1} = \sum_{w \in \mathrm{P}_{i,j}} wt(w)$$

Proof. We will proof this theorem by induction. By definition, we know that every entry in the quiddity row satisfies the formula:

$$m_{i-1,i+1} = \sum_{w \in \mathrm{P}_{i,i}} wt(w) = \sum_{p \in \mathrm{Poly}(v_i)} \lambda_{|\mathcal{P}|}$$

For every entry in row 0, which is the trivial row, we also have the formula:

$$m_{i-1,i} = 1 = wt(\text{length 0 path})$$

Suppose that all entries in the first k rows satisfy the argument.
For an arbitrary entry $m_{i,i+k+2}$ in row $k+1$, we have the recurrence

$$m_{i,i+k+2} = m_{i-1,i+1}m_{i+1,i+k+2} - m_{i+2,i+k+2}$$

Thus, it is suffice for us to proof that

$$\sum_{w \in \mathrm{P}_{i+1,i+k+1}} wt(w) = \sum_{w \in \mathrm{P}_{i+1,i+1}} wt(w) \sum_{w \in \mathrm{P}_{i+2,i+k+1}} wt(w) - \sum_{w \in \mathrm{P}_{i+3,i+k+1}} wt(w)$$

We will prove this by induction.
Now we will construct path from v_{i+1} to v_{i+k+1} by composing paths of shorter length.
Any path $w = (\mathcal{P}_{i+1}, \mathcal{P}_{i+2}, ..., \mathcal{P}_{i+k+1}) \in \mathbf{P}_{i+1,i+k+1}$ can be obtained by appending the subgon $\mathcal{P}_{i+1}$ in front of a subpath $w[i+2 : i+k+1] = (\mathcal{P}_{i+2}, \mathcal{P}_{i+3}, ..., \mathcal{P}_{i+k+1})$. Because each $\mathcal{P}_{i+s}$ is incident to v_{i+s}, we know that $\mathcal{P}_{i+1} \in \mathrm{Poly}(i+1)$ and $w[2 : k+1] \in \mathbf{P}_{i+2,i+k+1}$. Similarly, by appending any subgon incident to v_{i+1} in front of a path $w' \in \mathbf{P}_{i+2,i+k+1}$, we would obtain a path in $\mathbf{P}_{v_{i+1},v_{i+k+1}}$. Furthermore, it is clear that if $w_1, w_2 \in \mathbf{P}_{i+1,i+k+1}$ are different paths, then either $w_1[i+1 : i+1] \neq w_2[i+1 : i+1]$ or $w_1[i+2 : i+k+1] \neq w_2[i+2 : i+k+1]$ and vice versa.
Thus, there exists a bijection between $\mathbf{P}_{v_{i+1},v_{i+k+1}}$ and the product $\mathrm{Poly}(v_{i+1}) \times \mathbf{P}_{i+2,i+k+1}$.

By lemma 1.2.1, we know that there is exactly one subgon $\hat{\mathcal{P}}$ that incident to both v_{i+1} and v_{i+2}. Then for each path $h \in \mathbf{P}_{i+3,i+k+1}$, there is exactly one path $w_h \in \mathbf{P}_{i+1,i+k+1}$ such that

- $w_h[i+3 : i+k+1] = h$
- $w_h[i+1 : i+1] = w_h[i+2 : i+2]$

$\hat{\mathcal{P}}$ is the repeated subgon $w_h[i+1 : i+1] = w_h[i+2 : i+2] = \hat{\mathcal{P}}$, and $w_h = (\hat{\mathcal{P}}, \hat{\mathcal{P}}, h)$.
Notice that $\hat{\mathcal{P}} \in \mathbf{P}_{v_{i+1},v_{i+1}}$ and $(\hat{\mathcal{P}}, h) \in \mathbf{P}_{i+2,i+k+1}$.
Thus, we can rewrite

$$\mathbf{P}_i \times \mathbf{P}_{i+2,i+k+1}$$

as a new set

$$\left(\mathbf{P}_i \times \mathbf{P}_{i+2,i+k+1} - \left\{ \underset{h \in \mathrm{P}_{i+3,i+k+1}}{\left(\hat{\mathcal{P}}, (\hat{\mathcal{P}}, h)\right)} \right\}\right) \cap \mathbf{P}_{i+3,i+k+1}$$

by replacing each pair of $(\hat{\mathcal{P}}, (\hat{\mathcal{P}}, h))$ by h. Then there exists a bijection f from $\mathbf{P}_{i+1,i+k+1}$ to this set. For each path $w[i+1 : i+k+1] = (\mathcal{P}_{i+1}, \mathcal{P}_{i+2}, \ldots, \mathcal{P}_{i+k+1}) \in \mathbf{P}_{i+1,i+k+1}$,

- If $\mathcal{P}_{i+1} \neq \mathcal{P}_{i+2}$, $f(w[i+1 : i+k+1]) = (\mathcal{P}_{i+1}, w[i+2 : i+k+1])$
- If $\mathcal{P}_{i+1} = \mathcal{P}_{i+2}$, $f(w[i+1 : i+k+1]) = w[i+3 : i+k+1]$

By the definition of weighting on path, we know that

$$wt(w[i+1 : i+k+1]) = \begin{cases} \lambda_{|\mathcal{P}_{i+1}|} wt(w[i+2 : i+k+1]) & f(w) \notin \mathbf{P}_{i+3,i+k+1} \\ \lambda_{|\mathcal{P}_{i+1}|} wt(w[i+2 : i+k+1]) - wt(w[i+3 : i+k+1]) & f(w) \in \mathbf{P}_{i+3,i+k+1} \end{cases}$$

Therefore, we have

$$\begin{aligned}
\sum_{w \in \mathrm{P}_{i+1,i+k+1}} wt(w) &= \sum_{f(w) \notin \mathrm{P}_{i+3,i+k+1}} wt(w) + \sum_{f(w) \in \mathrm{P}_{i+3,i+k+1}} wt(w) \\
&= \left(\sum_{p \in \mathrm{P}_{i+1,i+1}} \sum_{w \in \mathrm{P}_{i+2,i+k+1}} wt(p) wt(w) - \sum_{f(w) \in \mathrm{P}_{i+3,i+k+1}} \lambda_{\hat{\mathcal{P}}} wt((\hat{\mathcal{P}}, f(w))) \right) \\
&+ \left(\sum_{f(w) \in \mathrm{P}_{i+3,i+k+1}} \lambda_{\hat{\mathcal{P}}} wt((\hat{\mathcal{P}}, f(w))) - wt(f(w)) \right) \\
&= \sum_{w \in \mathrm{P}_{i+1,i+1}} wt(w) \sum_{w \in \mathrm{P}_{i+2,i+k+1}} wt(w) - \sum_{h \in \mathrm{P}_{i+3,i+k+1}} \lambda_{\hat{\mathcal{P}}} wt((\hat{\mathcal{P}}, h)) \\
&+ \sum_{h \in \mathrm{P}_{i+3,i+k+1}} \lambda_{\hat{\mathcal{P}}} wt((\hat{\mathcal{P}}, h)) - \sum_{h \in \mathrm{P}_{i+3,i+k+1}} wt(h) \\
&= \sum_{w \in \mathrm{P}_{i+1,i+1}} wt(w) \sum_{w \in \mathrm{P}_{i+2,i+k+1}} wt(w) - \sum_{h \in \mathrm{P}_{i+3,i+k+1}} wt(h)
\end{aligned}$$

Thus, we completed the induction $m_{i,i+k+2} = \sum_{w \in \mathrm{P}_{i+1,i+k+1}} wt(w)$ as desired. □

Remark 3.0.1. Lemma 1.2.1 also applies to the case where $\overline{\mathcal{D}}$ is a quotient dissection on infinite strip. And all arguments in the proof of Theorem 3.0.1 hold true when $\mathcal{F}$ is realizable by a quotient dissection on annulus.

Proposition 3.0.1. Every nontrivial entry in an infinite frieze pattern of period n that correspond to a quotient dissection $\mathcal{D}$ satisfies that

$$m_{i-1,j+1} = \sum_{w \in \mathrm{P}_{i,j}} wt(w)$$

Recall that we can calculate the weight of a path by an explicit formula described in lemma 1.2.3, which is

$$wt(w_{i,j}) = \prod_{t=0}^{t=h} U_{a_{t+1}-a_t}(\lambda_{|\mathcal{P}_{a_t+1}|})$$

3.1 Skeletal dissections

Let $\mathcal{F}$ be a realizable skeletal frieze, with a corresponding skeletal dissection $\mathcal{D}$.
Any outer subgon in $\mathcal{D}$ contains at least 1 inner vertix, which are those incident to the two bridge arcs that are boundaries of the outer subgon. Thus, any outer subgon contains at most $p-1$ outer vertices, which appear consecutively on the annulus outer boundary. Any inner subgon is incident to exactly one outer vertices. Therefore, an identical subgon in a skeletal dissection must be chosen consecutively in any path. We can thus simplify the explicit formula for path weight by counting the number of times a subgon p is chosen in the path and use this number as the power of the Chebyshev polynomial being evaluated at $|\lambda_p|$.

We cannot choose any p-subgon in the skeletal dissection $\mathcal{D}$ consecutively more than $p-1$ times. By property of Chebyshev polynomials, we know that $U_k(\lambda_p) \geq 0$ when $k \in [p-1]$. Thus, we can determine the positivity of path weights:

Lemma 3.1.1. Any path in a skeletal dissection on annulus has a nonnegative weight.

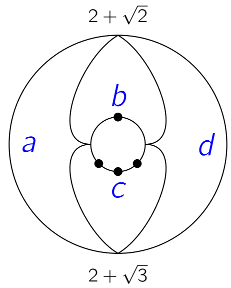
Corollary 3.1.1. All nontrivial entries in a realizable skeletal frieze are positive.

Remark 3.1.1. There are only a few paths that have weight 0 in skeletal dissections. We can always construct a positively weighted path from v_a to v_b by always choosing the outer subgon that contains the edge (v_i, v_{i+1}) for all $a \leq i \leq b$.

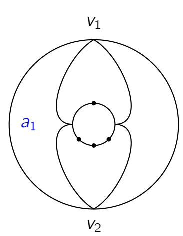
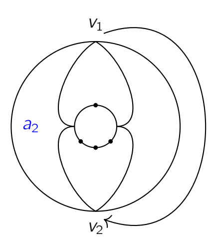
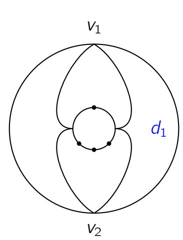
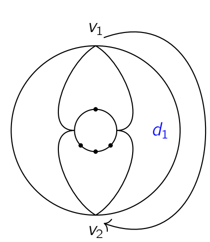
Corollary 3.1.2. Let $\mathcal{D}$ be a skeletal dissection on annulus and read off a quiddity sequence q from D. The frieze determined by q is a positive frieze.

Corollary 3.1.3. All nontrivial entries in a skeletal frieze of type Λ_p are positive.

Example 3.1.1. We will show all paths from vertex 1 to vertex 2, whose weights sum up to $m_{0,3} = 3 + 2\sqrt{2} + 2\sqrt{3} + \sqrt{6}$.



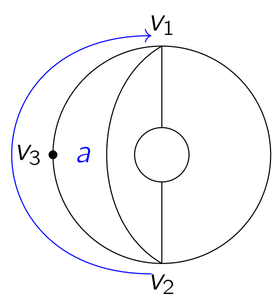
$$\begin{array}{cccccc}
\cdots & 0 & & 0 & & \cdots \\
\cdots & & 1 & & 1 & \cdots \\
\cdots & 2+\sqrt{2} & & 2+\sqrt{3} & & \cdots \\
\cdots & & 3+2\sqrt{2}+2\sqrt{3}+\sqrt{6} & & 3+2\sqrt{2}+2\sqrt{3}+\sqrt{6} & \cdots \\
 & & \iddots & & \ddots &
\end{array}$$

v_1	v_2	$wt(w)$
		$U_1(\lambda_3)U_1(\lambda_3) = 1$
a_1	c	$\sqrt{3}$
a_1	d	1
b	a_2	$\sqrt{2}$
b	c	$\sqrt{6}$
b	d	$\sqrt{2}$
d	a_2	1
d	c	$\sqrt{3}$
		$U_2(\lambda_3) = 0$
		$3+2\sqrt{2}+2\sqrt{3}+\sqrt{6}$

3.2 Non-skeletal dissections

If $\mathcal{D}$ is a dissection that contains peripherial arcs, then some p-subgon may be chosen up to p times in a path. If this p-subgon is an ear, then it can be chosen consecutively up to p times in a path, giving the path a $U_p(\lambda_p) = -1$ factor in its weighting.
If p is not an ear, but a subgon containing two peripheral arcs, then this exact subgon (meaning the infinite strip index is fixed) can be chosen nonconsecutively. The same subgon chosen nonconsecutively would be considered different in a path when calculating its weighting.

Example 3.2.1. $w = (a_1, a_1, a_1) \in \mathbf{P}_{2,4}$, $wt(w) = -1$.

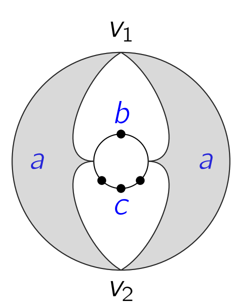


Conjecture 3.2.1. For every negatively weighted path, we can find a corresponding positively weighted path that cancels the negative term. Hence, every realizable infinite frieze is positive.

3.2.1 Quotient dissections

If a quotient dissection $\mathcal{D}$ is obtained by identifying all outer subgons in some skeletal dissection $\mathcal{D}_0$ as one p-gon $\hat{\mathcal{P}}$, then we can consecutively chose this $\hat{\mathcal{P}}$ as many times as we want in a path.

Example 3.2.2. $w = (a, a, a, \ldots, a, a, a) \in \mathbf{P}_{1,n}$, $wt(w) = U_n(1)$



4 Combinatorial interpretation of the growth coefficients

Definition 4.0.1. Let $\mathcal{D}$ be a skeletal dissection on a punctured disk, or an annulus with n vertices, then the *annulus weighting* of a path $w = (\mathcal{P}_i, \ldots, \mathcal{P}_j)$ is defined as:

$$wt_A(w) = \prod_{\text{Distinct } \mathcal{P} \text{ in } w} U_{N(\mathcal{P})}(\lambda_{|\mathcal{P}|})$$

where $N(p)$ is the number of times $\mathcal{P}$ is used in w.

Remark 4.0.1. Basically, the path weight definition we have used previously is calculating the weighting of a path on the infinite strip dissection, but the annulus weighting is calculating based on the annulus dissection.
Recall that when calculating the normal weighting of path in skeletal dissection, we can omit the consecutivity requirement for a fixed subgon in a skeletal dissection on the infinite strip can only be chosen consecutively. In annulus weighting, we also don't require a subgon to be chosen consecutively in order to be considered as choosing the same subgon.

Baur et al. determined the growth coefficient of infinite frieze pattern in [10].

Theorem 4.0.1 (Growth Coefficient Theorem, BFPT)**.** Given an infinite frieze pattern with period n, the growth coefficient $s_k := m_{i,i+kn+1} - m_{i+1,i+kn}$ is constant for each $k \geq 1$.

Theorem 4.0.2. Let $\mathcal{F}$ be a realizable skeletal infinite frieze pattern of period n and let s_1 denote its principal growth coefficient, then

$$s_1 = \sum_{w \in \mathrm{P}_{i+1,i+n}} wt_A(w)$$

Proof. First notice that for any i, a path in $\mathbf{P}_{i+1,i+n}$ would traverse all n outer vertices on the annulus.
By lemma 1.2.1, we know that there is exactly one subgon $\hat{\mathcal{P}}$ that contains the edge $(i+n, i+1)$.
We will first prove a claim: $wt((\mathcal{P}_{i+1}, \ldots, \mathcal{P}_{i+n})) \neq wt_A((\mathcal{P}_{i+1}, \ldots, \mathcal{P}_{i+n}))$ if and only if $p_{i+1} = p_{i+n} = \hat{\mathcal{P}}$.
By definition, $wt((\mathcal{P}_{i+1}, \ldots, \mathcal{P}_{i+n})) \neq wt_A((\mathcal{P}_{i+1}, \ldots, \mathcal{P}_{i+n}))$ if and only if there exists some $\mathcal{P}_t, \mathcal{P}_s$ in the path such that $\mathcal{P}_t$ and $\mathcal{P}_s$ are the same subgon on the annulus, but correspond to different copies of this subgon on the infinite strip.
If $\mathcal{P}_t$ is an inner subgon, then the next copy of it incident only to the outer vertex v_{n+t}, where $n + t > i + n$. Thus, if $\mathcal{P}_t$ is an inner subgon, there is no $\mathcal{P}_s$ in the path that would

correspond to a different copy of $\mathcal{P}_t$ on the infinite strip.
Suppose that $\mathcal{P}_t$ is an outer subgon, and that there is some $\mathcal{P}_s$ (without loss of generality, assume that $s > t$) in the path such that $\mathcal{P}_t$ and $\mathcal{P}_s$ are the same subgon on the annulus, but correspond to different copies of this subgon on the infinite strip.
There is a copy of $\mathcal{P}_s$ on the infinite strip that incident to vertex v_{s-n}. Because $s - t < n$, we know that this copy of $\mathcal{P}_s$ at v_{s-n} is the same copy of the subgon $\mathcal{P}_t$ at vertex v_t.
Because the dissection is skeletal, we know that this outer subgon contains all edges (v_{s-n}, v_{s-n+1}), (v_{s-n+1}, v_{s-n+2}),..., (v_{t-1}, v_t).
In particular, it contains the edge (v_{i+n}, v_i) on the annulus, meaning that this subgon is $\hat{\mathcal{P}}$ and $\mathcal{P}_t = \mathcal{P}_s = \hat{\mathcal{P}}$ on the annulus.
Either $t = i + 1$ and $s = i + n$ and we are done, or

- $t \neq i + 1$, then $\hat{\mathcal{P}}$ is the only subgon that incident to v_{i+1}, so $\mathcal{P}_{i+1} = \hat{\mathcal{P}}$.
- $s \neq i + n$, then $\hat{\mathcal{P}}$ is the only subgon that incident to v_{i+n}, so $\mathcal{P}_{i+n} = \hat{\mathcal{P}}$. □

Now we will focus on an arbitrary path $w \in \mathbf{P}_{i+1,i+n}$ such that its weighting on infinite strip and weighting on annulus are different. To emphasize that $\hat{\mathcal{P}}$ at v_{i+1} and $\hat{\mathcal{P}}$ at v_{i+n} are different copies on the infinite strip, we will denote them as $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$ respectively. A path on skeletal dissections on infinite strip can only choose the same copy consecutively, and we will use a denote the number of times $\hat{\mathcal{P}}$ has been used at the beginning of the path and use b denote the number of times $\hat{\mathcal{P}}'$ has been used at the ending of the path. We will use Q to denote the subpath obtained from w by deleting all of the $\hat{\mathcal{P}}$ and $\hat{\mathcal{P}}'$, i.e. $Q = w[a+1 : n-b]$.

$$\begin{aligned} wt(w) - wt_A(w) &= U_a(\lambda_{|\hat{\mathcal{P}}|})wt(Q)U_b(\lambda_{|\hat{\mathcal{P}}|}) - U_{a+b}(\lambda_{|\hat{\mathcal{P}}|})wt(Q) \\ &= U_{a-1}(\lambda_{|\hat{\mathcal{P}}|})U_{b-1}(\lambda_{|\hat{\mathcal{P}}|})wt(Q)(*) \\ &= wt(w[i+2 : i+n-1]) \end{aligned}$$

The $*$ step results from the equation $U_a(x)U(x_b) - U_{a-1}(x)U_{b-1}(x)$. We will prove this equation by induction.

$$\begin{aligned}
&U_a(x)U_b(x) - U_{a-1}(x)U_{b-1}(x)\\
=&U_1(x)U_{a-1}(x)U_b(x) - U_{a-2}(x)U_b(x) - U_{a-1}(x)U_{b-1}(x)\\
=&U_{a-1}(x)(U_1(x)U_b(x) - U_{b-1}(x)) - U_{a-2}(x)U_b(x)\\
=&U_{a-1}(x)U_{b+1}(x) - U_{a-2}(x)U_b(x)\\
=&U_1(x)U_{a-2}(x)U_{b+1}(x) - U_{a-3}(x)U_{b+1}(x) - U_{a-2}(x)U_b(x)\\
=&U_{a-2}(x)(U_1(x)U_{b+1}(x) - U_b(x)) - U_{a-3}(x)U_{b+1}(x)\\
=&U_{a-2}(x)U_{b+2}(x) - U_{a-3}(x)U_{b+1}(x)\\
&\vdots\\
=&U_1(x)U_{a+b-1}(x) - U_{a+b-2}(x)\\
=&U_{a+b}(x)
\end{aligned}$$

For each path $v \in \mathbf{P}_{i+2,i+n-1}$, there exists exactly a distinct path $w = (\hat{\mathcal{P}}, v, \hat{\mathcal{P}}')$ such that $wt(w) \neq wt_A(w)$ and that for every path $w \in \mathbf{P}_{i+1,i+n}$ such that $wt(w) \neq wt_A(w)$ there is a distinct path $v = w[i+2 : i+n-1] \in \mathbf{P}_{i+2,i+n-1}$ obtained by deleting one $\hat{\mathcal{P}}$ at both ends. Thus, we have

$$\sum_{w \in \mathrm{P}_{i+1,i+n}} wt(w) - \sum_{w \in \mathrm{P}_{i+1,i+n}} wt_A(w) = \sum_{w \in \mathrm{P}_{i+2,i+n-1}} wt(w)$$

Because

$$s_1 = \sum_{w \in \mathrm{P}_{i+1,i+n}} wt(w) - \sum_{w \in \mathrm{P}_{i+2,i+n-1}} wt(w)$$

, we have shown that

$$s_1 = \sum_{w \in \mathrm{P}_{i+1,i+n-1}} wt_A(w)$$

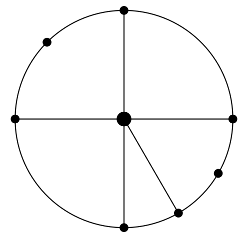
□

Corollary 4.0.1. Let $\mathcal{F}$ be a realizable skeletal infinite frieze pattern of period n with a corresponding dissection $\mathcal{D}$. Let s_k denote its k^{th} growth coefficient, then

$$s_k = \sum_{w \in \mathrm{P}_{i+1,i+kn+1} \text{ in } \mathcal{D}^k} wt_A(w)$$

Corollary 4.0.2. The growth coefficients of a skeletal frieze whose realizations are dissections on once-punctured disks are always 2.

Proof. A skeletal dissection on an once-puntured disk is always in the form of a *wheel*, that is, each vertex incidents with exactly 2 subgons.



In any polygon path that traverse all outer vertices of the once-punctured disk, the path has a non-zero annulus weight if and only if each subgon is chosen $p-2$ times. There are exactly 2 paths that satisfies this requirement: for all vertices that has 2 subgons incident to them, they either all choose the clockwise one, or all choose the counter clockwise one. Because $U_{p-2}(\lambda_p) = 1$, the growth coefficients of a skeletal frieze whose realizations are dissection on once-punctured disks are always 2. □

Remark 4.0.2. Tschabold has proven in [6] that all infinite friezes of positive integers that arise from triangulations on once punctured disk have growth coefficient 2 and vice versa.

5 Acknowledgment

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