Plan: We will use the combinatorial method of alcove walks to understand geometrically-interesting “cells” of matrix groups. (Intersection $\mathcal{U} \cap \mathcal{I} \cap \mathcal{I}_w$ of double cosets)

Part I: The algebra

1) The flag variety

A Lie group is a group that is also a manifold.

(locally like Euclidean space)
— They’re everywhere
  (connections to nearly every area of math & physics)

— Most Lie groups are matrix groups
  e.g. $GL_n, SL_n, SO_n, Sp_n$, over $\mathbb{R}$ or $\mathbb{C}$

— Beautiful, detailed structures

**Miracle:** much of the structure holds over any field ("Chevalley Groups")

For today: $G = SL_n$
(LET’S AGREE THAT SOME DEFINITIONS & ALL EXAMPLES WILL HAVE $G = SL_3$)
Let $B$ be the subgroup of upper triangular matrices (Borel subgroup):

$$B = \begin{bmatrix}
* & * & * \\
* & * & * \\
* & & 
\end{bmatrix}$$

Quotient $G/B$ : flag variety

A flag is a sequence of subspaces

$$\{0\} = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V$$

where $\dim V_i = i$.

Flag variety: one of the most important objects in algebra

However: $B$ is not normal, so $G/B$ is not a group!
Brilliant "fix": instead of left cosets, let's consider double cosets.

Given \( g \in G \), \( BgB = \{ g' \in G | g' = b_1gb_2, b_1, b_2 \in B \} \).

Double cosets are disjoint, so we can write:

**Bruhat decomposition:** \( G = \bigsqcup_{w \in W} BwB \)

Key fact: Turns out \( W \) is a group, called the Weyl group for \( G \).

(For \( G = SL_n \), \( W = S_n \)).

So, \( G/B = \bigsqcup_{w \in W} (BwB)/B \) (union of left \( B \) cosets)
Upshot: every element $gB$ of $G/B$ corresponds to a unique $w\in W$ and a (usually nonunique) $b\in B$: $gB = bwB$

#cool connection to Sunita's project: membership in double Bruhat cells $BwB$ gives a criterion for total positivity!

2) The affine flag variety

Going to step it up!

Field has been arbitrary up to now, but from now on, let

$$G = SL_n(F), \text{ where } F = \mathbb{C}((t))$$

$F$ is the fraction field of $O = \mathbb{C}[[t]]$. 
O has unique maximal ideal \((t)\), and there is a map \(O \to \mathbb{C}\) setting \(t = 0\).

E.g., \(1 + 2t + 3t^2 + 4t^3 + \ldots \to 1\)

This induces a map \(\text{SL}_n(O) \xrightarrow{\phi} \text{SL}_n(\mathbb{C})\).

Iwahori subgroup:

\[
I = \{ g \in \text{SL}_n(O) \mid \phi(g) \in B^2 \}
\]

\[
I = \begin{bmatrix}
0 & 0 & 0 \\
(t) & 0 & 0 \\
(t) & (t) & 0
\end{bmatrix}
\]

The affine flag variety is \(G/I\).

Again, not a group, but:
Iwahori decomposition:

\[ G = \bigsqcup_{\omega \in \tilde{W}} I_w I, \]

and \( \tilde{W} \) is a group, called the affine Weyl group.

Example: Let \( g = \begin{bmatrix} \frac{1}{t} & 2t & 2t^2 \\ t & t^2 & 1 \end{bmatrix} \)

Then \( g \in B \), so

\[ g \in B \cdot e_B \cdot e_W \cdot e_B \]

\( g \in B1B \)
Also,

\[
g = \begin{bmatrix} \frac{1}{t} & 2t \\ t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}
\]

Notice that the elements of \( W \) are the same.

Now, \( g \neq I \), but

\[
g = \begin{bmatrix} 1 & 2 & 2t^2 \\ 1 & t^2 & t \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} t^{-1} \\ t \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

Now, let's explore \( W, \tilde{W} \).
3) Weyl group & affine Weyl group

Let $G = \text{SL}_3$, so $W = S_3$, $\tilde{W} = \tilde{S}_3$

Note that $s_1 = (12), s_2 = (23) \in S_3$ have order 2.

$$S_3 = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1, s_1s_2s_1 = s_2s_1s_2 \rangle$$

(Coxeter presentation)

Pictorially:

- $s_1$-edge
- $s_2$-edge

$\Delta = \text{alcove}$

Similarly,

$$\tilde{S}_3 = \langle s_0, s_1, s_2 \mid s_0^2 = s_1^2 = s_2^2 = 1, s_0s_1s_0 = s_1s_0s_1, s_0s_2s_0 = s_2s_0s_2, s_1s_2s_1 = s_2s_1s_2 \rangle$$
REU Exercise 7.1

a) Write out all 6 elements of $S_3$ as minimal length products of $s_1, s_2$. What is special about (13)?
b) Prove that $S_3$ bijects with the alcoves in the first diagram.

c) Prove that $\tilde{S}_3$ bijects with the alcoves in the second diagram. You just proved that $\tilde{S}_3$ is infinite!

4) Steinberg generators

First another decomposition:

Let $U^{-} = \begin{bmatrix} 1 & \ast & \ast \\ \ast & 1 & \ast \\ \ast & \ast & 1 \end{bmatrix}$. Then,

$$G = \bigsqcup_{\omega \in \tilde{W}} U^{-} \omega I$$

affine Weyl group
Let's get more precise information about the elements of $U^-, I, \tilde{W}$

Steinberg generators:

\[
\begin{align*}
X_1(c) &= \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} & X_{-1}(c) &= \begin{bmatrix} 1 & c \\ -c & 1 \end{bmatrix} \\
X_2(c) &= \begin{bmatrix} 1 & c \\ c & 1 \end{bmatrix} & X_{-2}(c) &= \begin{bmatrix} 1 & c \\ -c & 1 \end{bmatrix} \\
X_0(c) &= \begin{bmatrix} 1 & ct \\ ct & 1 \end{bmatrix} & X_{-0}(c) &= \begin{bmatrix} 1 & ct \\ -ct & 1 \end{bmatrix}
\end{align*}
\]

Let $n_i(c) = x_i(c) x_{-a_i}(-c^{-1}) x_i(c)$,

\[
\begin{align*}
n_i : &= n_i(1), & h_i(c) &= n_i(c) n_i^{-1}
\end{align*}
\]
REV Exercise 7.2:

1. **a)** Show that \( x_i(c_1)x_i(c_2) = x_i(c_1+c_2) \)

2. **b)** Compute \( n_i, h_i(c), \ i = 0, 1, 2 \)
   Which of the \( x_0, n_i, h_i \) are in \( U^- \)?
   Which are in \( I \)?

3. **c)** Prove that (up to flipping signs)
   \( n_0, n_1, n_2 \) satisfy the same relations as \( s_0, s_1, s_2 \)

4. **d)** Solve the following equation for \( i, j = 0, 1, 2 \):
   \( n_i^{-1}x_j(c) = x_i(\ ? \ ) \ldots x_i(\ ? \ )n_i^{-1} \)

5. **e)** Prove symbolically that if \( c \neq 0 \)
   \( X_i(c)n_i^{-1} = X_{-a_i}(c^{-1})X_i(-c)h_i(c) \)

6. **f)** Use parts d, e to show that when \( j \neq i \),
   \( n_j^{-1}x_i(c)n_i^{-1} \in U^- n_j^{-1} I \).
**Part II: The alcove walk model**

\[
U^{-v} I = \left\{ x_{y_1}(d_1) \cdots x_{y_k}(d_k) n_{j_1}^{-1} \cdots n_{j_k}^{-1} I \bigg| d_1, \ldots, d_k \in \mathbb{C} \right\} \\
(v \in \hat{W})
\]

\[
\in U^- \quad v = s_{j_1} \cdots s_{j_k}
\]

**Theorem 1 (Parkinson–Ram–Schwer ’08):**

Let \( w = s_{i_1} \cdots s_{i_k} \in \hat{W} \) be a reduced expression. Then in \( G/I \),

\[
I w I = \left\{ x_{i_1}(c_1) h_{i_1}^{-1} \cdots x_{i_k}(c_k) h_{i_k}^{-1} I \bigg| c_1, \ldots, c_k \in \mathbb{C} \right\}
\]

1) Alcove walks
(Labelled) alcove walk: A shortest path walk to \( w \), where every edge is labelled by an element of \( \mathbb{C} \).

Corollary (PRS '08):

\[
\left\lfloor \frac{IwI}{I} \right\rfloor \leftrightarrow \{ \text{labelled alcove walks from } 1 \text{ to } w \}
\]

2) Folded alcove walks

Let the "sun" be at the top of the page. The positive side of each edge is the side that the sun hits.

We look at positively-folded alcove walks: (edge-labels are implied)
This is a positively folded alcove walk of type $w$ ending in $v$. 

$w = s_1 s_2 s_0 s_2 s_1$

$v = s_1 s_0 s_2 s_1$
Theorem 2 (PRS '08): In $G/I$, there is a bijection:

$$\frac{(U^-vI \cap IwI)}{I} \leftrightarrow \left\{ \text{labelled positively folded alcove walks of type } w \text{ which end in } v \right\}$$

Proof technique: Apply the main folding law repeatedly to an element of $IwI$.

REU Exercise 7.3: Let $w = s_2 s_1 s_0 s_1 s_2, v = s_2 s_0 s_1 s_2$

(a) How many alcove walks of type $w$ are there?
(b) Describe the elements of $IwI$. (Use Thm 1).
(c) How many positively folded alcove walks of type $w$ ending in $v$ are there?
(d) Describe the elements of $U^-vI \cap IwI$ using (b), (c), Thm 2, and the following label restrictions:
3) Triple intersections

Theorem 3 (PRS, Beazley - Brubaker):

a) \( U^+ vI \cap I_wI \leftrightarrow \begin{cases} \text{labelled negatively folded} \\ \text{alocove walks of type } w \\ \text{ending in } V \end{cases} \)

b) The triple intersection

\( U^- v, I \cap I_wI \cap U^+ v_2I \leftrightarrow \begin{cases} \text{labelled positively folded} \\ \text{alocove walks of type } w \text{ ending in } v_1 \text{ that} \\ \text{correspond to negatively} \\ \text{folded alcove walks ending} \\ \text{in } v_2. \end{cases} \)
Theorem 4 (Beazley-Brubaker): When $G = \text{SL}_2$, the above bijection allows us to evaluate a certain number theoretic "special function" on $\text{SL}_2$ in terms of Gelfand-Tsetlin patterns. (Cool connection to Ben's project)

REU Problem 7: (Also: algebraic interpretation of the sun).

a) For $G = \text{SL}_3$, given $w, v_1, v_2 \in \tilde{W}$, when is $U^{-v_1}I \cap IwI \cap U^{v_2}I$ nonempty?

b) Figure out a combinatorial formula for its size (i.e. measure)

c) Can we do the same thing for other Chevalley groups ($\text{SL}_4$, $\text{SL}_n$, $\text{GL}_n$), or for other double coset decompositions?

d) Can we use our results on triple intersections to compute certain special functions on $G$?