Whittaker functions and the alcove walk model

Neelima Borade, Matthew Huynh, Henry Twiss

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Outline

1. Whittaker Functions
2. The Alcove Walk Model
3. An Introduction to Folding
4. References
Whittaker Functions

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References
Where Whittaker Functions Appear

- Special functions over a split reductive group $G(F)$. 

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- Special functions over a split reductive group $G(F)$.
- Historically, they referred to a solution to a confluent hypergeometric equation proposed by Whittaker.
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• Basic tool in aut. forms and the construction of $L$-functions.
Where Whittaker Functions Appear

- Special functions over a split reductive group $G(F)$.
- Historically, they referred to a solution to a confluent hypergeometric equation proposed by Whittaker.
- Basic tool in automorphic forms and the construction of $L$-functions.
- Arise as common eigenfunctions in physics.
The Whittaker Model

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\[ G(F) \text{ is split reductive: torus } T, \text{ unipotent } U, \text{ Borel } B = TU, \]
$G(F)$ is split reductive: torus $T$, unipotent $U$, Borel $B = TU$, maximal compact $K$. 
The Whittaker Model

$G(F)$ is split reductive: torus $T$, unipotent $U$, Borel $B = TU$, maximal compact $K$. The Whittaker model is the space of functions $W$ satisfying

$$W(ug) = \psi(u)W(g) \text{ (Whittaker relation).}$$
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$G(F)$ acts by right translation.
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**Theorem 1.1 (Gelfand-Grave, Jaquet-Langlands, etc. al)**

Given an irr. rep. $(\pi, V)$ of $G(F)$ there is at most one isomorphic copy inside the Whittaker model under this action by right-translation.
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Upshot: We view $(\pi, V)$ as functions on $G(F)$. 

The Spherical Function

Inside \((\pi, V)\) there is a spherical vector \(\phi_K\) fixed by the action of \(K\).
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Inside $(\pi, V)$ there is a spherical vector $\phi_K$ fixed by the action of $K$. Inside the Whittaker model $\phi_K$ is

$$W(t^\lambda) = \int_{U^-} v_K(ut^\lambda)\psi(u)\,du.$$
The Spherical Function

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W(t^\lambda) = \int_{U^-} v_K(ut^\lambda)\psi(u)\,du.
\]

This is the (spherical) Whittaker function.
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The Alcove Walk Model for $\mathfrak{s}_3$
The Alcove Walk Model for $\mathfrak{sl}_3$

\[ H_{\alpha_2 + \delta} \quad H_{\alpha_2} \quad H_{-\alpha_2 + \delta} \]

\[ \cdots \quad - \quad + \quad - \quad + \quad \cdots \]

\[ H_{\alpha_1 + \delta} \quad H_{\alpha_1} \quad H_{-\alpha_1 + \delta} \]

\[ s_2 \quad s_1 \quad s_2 s_1 \quad s_1 s_2 \]

\[ H_{\alpha_0} \quad H_{\varphi} \quad H_{\varphi + \delta} \]

\[ w_0 \]
The Setting
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We work in a Euclidean space $h^*_R$. 
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We work in a Euclidean space $\mathfrak{h}^*_\mathbb{R}$. Inner product $(\cdot, \cdot)$. Set

$$\langle \alpha, \beta \rangle = \frac{2(\alpha, \beta)}{(\alpha, \alpha)}.$$
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The affine Weyl group:

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\[ W := \langle s_i \mid 1 \leq i \leq n \rangle \]

is the finite Weyl group.
The Alcove Walk Model for $\mathfrak{sl}_3$
The Alcove Walk Model for $\mathfrak{sl}_3$
The affine hyperplanes are

\[ H_{\alpha_i + j\delta} := \{ \beta \in \mathfrak{h}_\mathbb{R}^* \mid \langle \alpha_i, \beta \rangle = j \}. \]
Hyperplanes and Reflections

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The affine reflection over \( H_{\alpha_i + j\delta} \) is

\[ s_{\alpha_i + j\delta}(\beta) := \beta - (\langle \alpha, \beta \rangle + j)\alpha_i. \]
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\( \{\alpha_1, \ldots, \alpha_n\} \) form a basis for a root system. Think of \( \alpha_0 \) as the highest root \( \varphi \) (highest weight of adj. rep.).

\[ W_{\text{aff}} = \langle s_i := s_{\alpha_i} \mid 0 \leq i \leq n \rangle. \]
The Case for $\mathfrak{sl}_3$
The Case for $\mathfrak{sl}_3$
The Case for \( \mathfrak{sl}_3 \)

\[
\begin{align*}
W & \cong S_3 \quad \text{and} \quad W_{\text{aff}} \cong \tilde{S}_3.
\end{align*}
\]
The Case for $\mathfrak{sl}_3$

The highest root is

$$\varphi = \alpha_1 + \alpha_2.$$
The Alcove Walk Model for $\mathfrak{s}_{\frac{1}{3}}$
The Alcove Walk Model for \( \mathfrak{sl}_3 \)

\[ H_{\alpha_2 + \delta} \quad H_{\alpha_2} \quad H_{-\alpha_2 + \delta} \]

\[ \begin{align*}
H_{\alpha_1 + \delta} & \quad H_{\alpha_1} & \quad H_{\alpha_1 + \delta} \\
- & \quad + & \quad - & \quad + & \quad - & \quad + \\
+ & \quad - & \quad + & \quad - & \quad + & \quad - \\
\end{align*} \]
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The Coroot Lattice
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$$Q^\vee := \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$$

where

$$\alpha_i^\vee := \frac{2\alpha_i}{(\alpha_i, \alpha_i)}.$$
The Coroot Lattice

The centers of hexagons are in bijective correspondence with $Q^\vee$:

$$Q^\vee := \mathbb{Z}\alpha_1^\vee + \cdots + \mathbb{Z}\alpha_n^\vee$$

where

$$\alpha_i^\vee := \frac{2\alpha_i}{(\alpha_i,\alpha_i)}.$$  

Also,

$$W_{\text{aff}} \cong W \rtimes Q^\vee$$

under translation by $Q^\vee$. 
The Case for $\mathfrak{sl}_3$
The Case for $\mathfrak{sl}_3$
The Case for $\mathfrak{sl}_3$

\[ Q^\vee = \mathbb{Z}\alpha_1^\vee + \mathbb{Z}\alpha_2^\vee \quad \text{and} \quad \tilde{S}_3 \cong S_3 \rtimes Q^\vee. \]
The Alcove Walk Model for $\mathfrak{s}_3$
The Alcove Walk Model for $\mathfrak{sl}_3$
• 1 lies on the positive side of the $H_{\alpha_i}$
Hyperplane Orientation

- 1 lies on the positive side of the $H_{\alpha_i}$
- $H_{\alpha_i+j\delta}$ and $H_{\alpha_i}$ have parallel orientations.
Hyperplane Orientation

- 1 lies on the positive side of the $H_{\alpha_i}$
- $H_{\alpha_i + j\delta}$ and $H_{\alpha_i}$ have parallel orientations.

These facts dictate most of the combinatorics about the walk.
The Alcove Walk Model for $\mathfrak{s}_3$
The Alcove Walk Model for $\mathfrak{sl}_3$

\[ H_{\alpha_2 + \delta} \quad H_{\alpha_2} \quad H_{-\alpha_2 + \delta} \]

\[ H_{\alpha_0} \quad H_{\varphi} \quad H_{\varphi + \delta} \]

\[ s_2, \quad s_1, \quad w_0, \quad s_2 s_1, \quad s_1 s_2 \]
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An Alcove Walk

The walk for $w = s_1s_2s_0s_1$ is

\[
H_{\alpha_2+\delta} \quad H_{\alpha_2} \quad H_{-\alpha_2+\delta}
\]

\[
H_{-\alpha_1+\delta} \quad H_{\alpha_1} \quad H_{\alpha_1+\delta}
\]

\[
+ \quad - \quad + \quad - \quad + \quad -
\]

\[
+ \quad H_{\alpha_0} \quad - \quad + \quad \varphi \quad - \quad + \quad H_{\varphi_+\delta}
\]

\[
+ \quad H_{\varphi} \quad - \quad + \quad \varphi \quad - \quad + \quad H_{\varphi+\delta}
\]

\[
\varphi \quad - \quad \varphi \quad - \quad \varphi \quad - \quad \varphi \quad - \quad \varphi
\]
An Alcove Walk

The walk for $w = s_1s_2s_0s_1$ is

\[ H_{\alpha_2+\delta} \quad H_{\alpha_2} \quad H_{-\alpha_2+\delta} \]

\[ + \quad - \quad + \quad - \quad + \]

\[ \ldots \quad \ldots \quad \ldots \quad \ldots \]

\[ H_{\alpha_0}, \quad H_{\varphi}, \quad H_{\varphi+\delta}, \quad H_{-\alpha_1+\delta}, \quad H_{\alpha_1}, \quad H_{\alpha_1+\delta} \]
An Alcove Walk

The walk for $w = s_1 s_2 s_0 s_1$ is

\[
\begin{align*}
H_{\alpha_2 + \delta} & \quad H_{\alpha_2} & \quad H_{-\alpha_2 + \delta} \\
- & \quad + & \quad - & \quad + \\
| & \quad | & \quad | & \quad |
\end{align*}
\]
An Alcove Walk

The walk for \( w = s_1 s_2 s_0 s_1 \) is

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\begin{align*}
H_{\alpha_2 + \delta} & \quad H_{\alpha_2} & \quad H_{-\alpha_2 + \delta} \\
- & \quad + & \quad - & \quad + & \quad +
\end{align*}
\]

\[
\begin{align*}
H_{-\alpha_1 + \delta} & \quad H_{\alpha_1} & \quad H_{\alpha_1 + \delta} \\
+ & \quad - & \quad + & \quad - & \quad -
\end{align*}
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The walk for $w = s_1 s_2 s_0 s_1$ is
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We label each step of the walk with elements in a finite field $\mathbb{F}_q$. 
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$$|wl| \leftrightarrow \{ \text{walks to } w \text{ with labels in } \mathbb{F}_q \}.$$
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Folding

Turns steps of the form

\[ H_{\pm \alpha_i + j\delta} \]

or

\[ H_{\pm \alpha_i + j\delta} \]
Turns steps of the form

\[ H_{\pm \alpha_i + j\delta} \]

into

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Folding

Turns steps of the form

\[ H_{\pm \alpha_i + j\delta} \]

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\[ H_{\pm \alpha_i + j\delta} \]

or

\[ H_{\pm \alpha_i + j\delta} \]

If \( c = 0 \) we cannot fold and if \( c \neq 0 \) we must fold!
A Folded Walk

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Let $w = s_1 s_2 s_0 s_1$ with labels $(0, 0, c, 0)$. 
Let $w = s_1 s_2 s_0 s_1$ with labels $(0, 0, c, 0)$.
Let $w = s_1 s_2 s_0 s_1$ with labels $(0, 0, c, 0)$. The folded walk is

$$H_{\alpha_2 + \delta} \quad H_{\alpha_2} \quad H_{-\alpha_2 + \delta}$$

$$H_{\alpha_1 + \delta} \quad H_{\alpha_1} \quad H_{\alpha_1 + \delta}$$

$$H_{-\alpha_1 + \delta}$$

$$H_{\varphi + \delta}$$

$$H_{\varphi}$$

$$H_{\alpha_0}$$
Folded Walks and Double Cosets

Two facts from Parkinson-Ram-Schwer:
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\[
\begin{align*}
\left\{ \text{labeled walks to } w \text{ that positively fold to } v_1 \right\} & \leftrightarrow U^- v_1 I \cap lwI.
\end{align*}
\]
Folded Walks and Double Cosets

Two facts from Parkinson-Ram-Schwer:

\[
\begin{align*}
\{ \text{labeled walks to } w \text{ that positively fold to } v_1 \} & \longleftrightarrow U^+ v_1 l \cap lw_1. \\
\{ \text{labeled walks to } w \text{ that negatively fold to } v_2 \} & \longleftrightarrow U^- v_2 l \cap lw_l.
\end{align*}
\]
Folded Walks and Double Cosets

Two facts from Parkinson-Ram-Schwer:

\[
\{ \text{labeled walks to } w \text{ that positively fold to } v_1 \} \longleftrightarrow U^- v_1 l \cap lw l.
\]

\[
\{ \text{labeled walks to } w \text{ that negatively fold to } v_2 \} \longleftrightarrow U^+ v_2 l \cap lw l.
\]

From Beazley-Brubaker:

\[
\{ \text{labeled walks to } w \text{ that positively fold to } v_1 \text{ and negatively fold to } v_2 \} \longleftrightarrow U^- v_1 l \cap lw l \cap U^+ v_2 l.
\]
Folded Walks and Double Cosets

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\[
\{ \text{labeled walks to } w \text{ that negatively fold to } v_2 \} \longleftrightarrow U^+ v_2 l \cap lw l.
\]

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\[
\left\{ \begin{array}{l}
\text{labeled walks to } w \\
\text{that positively fold to } v_1 \\
\text{and negatively fold to } v_2
\end{array} \right\} \longleftrightarrow U^- v_1 l \cap lw l \cap U^+ v_2 l.
\]

This last bijection makes our Whittaker functions extremely computable!
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