\textit{q-Analogues of Rational Numbers}

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Minnesota REU
Outline

1. *q*-Analogues
2. Definition
3. How to Compute
4. What Does it Count?
5. Cluster Algebras
6. *q*-Real Numbers
The $q$-Integers

Definition

For each $n \in \mathbb{N}$, define the polynomial $[n]_q \in \mathbb{Z}[q]$:

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$$

Remark: Substituting $q = 1$ gives $n$. 

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The $q$-Integers

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Rational Numbers

Question:

What about \([x/q]\) for \(x \in \mathbb{Q}\)? Can we make sense of this?

There are conceivably many definitions that "work". The question is what properties do we want it to satisfy?

Here are two properties that might be desirable:

Order:

Define a partial order on rational functions by

\[
\frac{a}{q} > \frac{c}{d}\]

if \(a d - b c\) has all positive coefficients. If \(\frac{a}{b} > \frac{c}{d}\), we might expect \(\frac{a}{b} q > \frac{c}{d} q\).

Convergence:

If \(a_n b_n \rightarrow \lambda \in \mathbb{R}\) irrational, we might expect \(a_n b_n q\) to "converge" in some sense, and moreover be independent of the sequence.
Rational Numbers

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- **Order:** Define a partial order on rational functions by \(\frac{a(q)}{b(q)} > \frac{c(q)}{d(q)}\) if \(a(q)d(q) - b(q)c(q)\) has all positive coefficients.
  If \(\frac{a}{b} > \frac{c}{d}\), we might expect \([\frac{a}{b}]_q > [\frac{c}{d}]_q\).
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  \]
  If \(\frac{a}{b} > \frac{c}{d}\), we might expect \(\left[\frac{a}{b}\right]_q > \left[\frac{c}{d}\right]_q\).

- **Convergence:** If \(\frac{a_n}{b_n} \rightarrow \lambda \in \mathbb{R}\) irrational, we might expect \(\left[\frac{a_n}{b_n}\right]_q\) to “converge” in some sense, and moreover be independent of the sequence.
First Naive Attempt

A first natural guess might be to define
\[
\begin{align*}
[a]_{q} &= [a]_{q}[b]_{q} = 1 + [a]_{q} + [a]_{q}[a]_{q} + \cdots + [a]_{q}^{a-1} - 1 + [b]_{q} + [b]_{q}[b]_{q} + \cdots + [b]_{q}^{b-1}
\end{align*}
\]

But... it does not satisfy our two desirable properties.

Exercise 9.1: Find an example where this definition does not satisfy the order property. That is, find two fractions \( \frac{a}{b} > \frac{c}{d} \) where \([a]_{q}[d]_{q} - [b]_{q}[c]_{q}\) has some negative coefficient.
A first natural guess might be to define

\[
\left[ \frac{a}{b} \right]_q := \frac{[a]_q}{[b]_q} = \frac{1 + q + \cdots + q^{a-1}}{1 + q + \cdots + q^{b-1}}
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Continued Fractions

A continued fraction is an expression consisting of nested fractions, like this:

\[ a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots + \frac{1}{a_n}}}} \]

We use the notation \([a_1, a_2, ..., a_n]\) to denote the expression above.

Example:

\[ \frac{7}{4} = 1 + \frac{1}{1 + \frac{1}{3}} \]. So we'd write
\[ \frac{7}{4} = [1, 1, 3] \].

Remark:
These are not unique. For example, \( \frac{7}{4} \) is also equal to \([1, 1, 2, 1]\). Requiring an even number of coefficients makes it unique.
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**Remark:** These are not unique. For example, \(\frac{7}{4}\) is also equal to \([1, 1, 2, 1]\). Requiring an even number of coefficients makes it unique.
Definition of $q$-Rationals

If $r/s = [a_1, a_2, \ldots, a_{2^n}]$, then define $[r/s]_q := [a_1]_q + q[a_2]_q - 1 + q-1a_2 \cdot \cdot \cdot + q[a_2^n]_q - 1$.

Example: $\frac{7}{3} = [2, 3]$. 

$[7/3]_q = 1 + q + q^2 + q^3 + q^4 + q^5 = 1 + 2q + 2q^2 + q^3 + q^4 + q^5$.

Fact: The only time this agrees with the "naive guess" is for $[n+1/n]_q = [n+1]_q [n]_q$.
Definition of $q$-Rationals

**Definition**

If $\frac{r}{s} = [a_1, a_2, \ldots, a_{2n}]$, then define

$$\left[ \frac{r}{s} \right]_q := [a_1]_q + \frac{q^{a_1}}{[a_2]_q^{-1} + \frac{q^{-a_2}}{[a_3]_q^{-2} + \cdots + \frac{q^{-a_{2n-1}}}{[a_{2n}]_q^{-2n-1}}}}$$

**Example:**

$7_3 = [2, 3]$. Thus,

$$\left[ \frac{7}{3} \right]_q = 1 + q + q^2 1 + q + q^3 + q^4 1 + q + q^2 + q^3 + q^4 + q^5 + q^6$$

**Fact:**

The only time this agrees with the "naive guess" is for $\left[ \frac{n+1}{n} \right]_q = \left[ \frac{n+1}{n} \right]_q$. 

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**Fact:** The only time this agrees with the “naive guess” is for $[\frac{n+1}{n}]_q = [\frac{[n+1]}{n}]_q$. 
The Desirable Properties
This definition of $\left[ \frac{a}{b} \right]_q$ does satisfy the order and convergence properties.
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Given \( r_s = [a_1, a_2, \ldots, a_n] \), we construct a triangulated polygon \( T_{r/s} \):

Example: \( 7/3 = [2, 3] \)

\( T_{7/3} = \) N.O. (UMN)
A Combinatorial Method of Computation

Given $\frac{r}{s} = [a_1, a_2, \ldots, a_n]$, we construct a triangulated polygon $T_{r/s}$:

\[ a_2 \]

\[ a_1 \]

\[ a_3 \]
A Combinatorial Method of Computation

Given \( \frac{r}{s} = [a_1, a_2, \ldots, a_n] \), we construct a triangulated polygon \( T_{r/s} \):

\[
\begin{array}{c}
\text{a_2} \\
\hline
a_1 \\
\hline
a_3 \\
\hline
\text{\ldots}
\end{array}
\]

Example: \( \frac{7}{3} = [2, 3] \)

\[
T_{7/3} = \]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{triangulated_polygon.png}
\end{figure}
A Combinatorial Method of Computation

Definition

The Farey sum of two rational numbers \( \frac{a}{b} \oplus \frac{c}{d} \) is \( \frac{a+c}{b+d} \).

Label the left two vertices of \( T/\sigma \) by 0 1 and 1 0. Going left to right, for each triangle, label the third vertex as the Farey sum of the previous two.

Example:

\[
\begin{array}{cccccc}
0 & 1 & 1 & 2 & 3 & 5 \\
3 & 7 & 4 & 1 & 5 & 2
\end{array}
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Label the left two vertices of \( T_{r/s} \) by \( \frac{0}{1} \) and \( \frac{1}{0} \).
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**Example:** \( \frac{7}{3} \)

\[
T_{7/3} = \begin{array}{c}
\frac{1}{0} \\
\frac{1}{1} \\
\frac{2}{1}
\end{array}
\]
Definition

The *Farey sum* of two rational numbers is

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Example: \( \frac{7}{3} \)

\[ T_{7/3} = \]

\[
\begin{array}{c}
\frac{1}{0} \quad \frac{3}{1} \quad \frac{5}{2} \\
\frac{2}{1} \quad \frac{1}{1}
\end{array}
\]
Definition

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Label the left two vertices of \( T_{r/s} \) by \( \frac{0}{1} \) and \( \frac{1}{0} \).

Going left to right, for each triangle, label the third vertex as the Farey sum of the previous two.

**Example:** \( \frac{7}{3} \)

\[
T_{7/3} = \frac{0}{1} \oplus \frac{1}{0} \oplus \frac{2}{1} \oplus \frac{3}{1} \oplus \frac{5}{2} \oplus \frac{7}{3}
\]
For each top vertex in $T_{rs}$, label the diagonals with increasing powers of $q$ going counter-clockwise:

Example:
The $q$-Version

For each top vertex in $T_{r/s}$, label the diagonals with increasing powers of $q$ going counter-clockwise:

$$\begin{array}{c}
q^3 \\
q^2 \\
q \\
1
\end{array}$$
For each top vertex in $T_{r/s}$, label the diagonals with increasing powers of $q$ going counter-clockwise:

Example: $\frac{7}{3}$
The $q$-Version

As before, start by labeling the left two vertices by $0$ and $1$. In each triangle, the two edges incident to the third vertex will be labelled by $1$ and $q^k$ for some $k$. Label the third vertex by the weighted Farey sum:

$$\frac{1}{q^k a + b} \frac{1}{q^k c + d}.$$

Example:

$$\left[ \begin{array} {cc} 7 & 3 \\ q & 1 \\ q & q \\ q & 2 \\ q & q \\ q & q \\ q & q \\ q & q \end{array} \right]$$
The $q$-Version

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\[
\begin{array}{c}
\frac{c}{d} \\
q^k \\
\frac{a}{b} \\
1 \\
\frac{a+q^k c}{b+q^k d}
\end{array}
\]
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\]

**Example:** $\left[ \frac{7}{3} \right]_q$
The \(q\)-Version

As before, start by labeling the left two vertices by \(\frac{0}{1}\) and \(\frac{1}{0}\). In each triangle, the two edges incident to the third vertex will be labelled by 1 and \(q^k\) for some \(k\). Label the third vertex by the weighted Farey sum:

\[
\frac{a}{b} \quad \frac{c}{d} \quad q^k
\]

\[
\frac{a + q^k c}{b + q^k d}
\]

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As before, start by labeling the left two vertices by $\frac{0}{1}$ and $\frac{1}{0}$.

In each triangle, the two edges incident to the third vertex will be labelled by $1$ and $q^k$ for some $k$. Label the third vertex by the weighted Farey sum:

$$\frac{a}{b} + q^k \frac{c}{d} = \frac{a + q^k c}{b + q^k d}.$$

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\]

Example: $\left[\frac{7}{3}\right]_q$
Exercise 9.2

(a) What does \( T_s \) look like for \([1, 1, ... , 1]\)?

(b) Prove that \([1, 1, ... , 1]\) is always a ratio of Fibonacci numbers.

(c) Use the triangulation method to compute \([5, 3, 0]\) and \([8, 5, 0]\).
Exercise 9.2:

(a) What does $T_{r/s}$ look like for $[1, 1, \ldots, 1]$?
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From Triangulations to Binary Words

From the triangulation $T$, we construct a binary word in the alphabet $\{R, U\}$ as follows. Ignore the first and last triangles. For the others, label their boundary edges $U$ if they are on the bottom, and $R$ if they are on the top.

Example: $T_7^3 \quad \begin{array}{c}
R
\end{array}$
From the triangulation \( T_{r/s} \), we construct a binary word in the alphabet \( \{R, U\} \) as follows.

Example:
\[
\begin{array}{cccc}
    & & & \\
    & & & \\
    & & & \\
    & & & \\
    & & & \\
    & & & \\
\end{array}
\]

N.O. (UMN)
From the triangulation $T_{r/s}$, we construct a binary word in the alphabet \{R, U\} as follows.

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**Example:** $T_{7/3}$
From Binary Words to Snake Graphs

From a binary word, we construct a graph $G_{r/s}$, called a snake graph, as follows.

Start with a square. For each letter in the binary word, add another square either above (for $U$) or to the right (for $R$) of the previous.

Example: $73$ has binary word $URR$. So the snake graph looks like $G_{7/3}$.

Caution: In the literature, “snake graph” refers to a slightly different, but related, construction. The construction above is called the “dual snake graph” corresponding to a triangulation.
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G_{7/3} =
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**Example:** $7_3$ has binary word $URR$. So the snake graph looks like

```
G_{7/3} =
```

**Caution:** In the literature, “snake graph” refers to a slightly different, but related, construction. The construction above is called the “dual snake graph” corresponding to a triangulation.
If $G$ is a snake graph, let $L(G)$ be the set of all paths in $G$ from the south-west corner to the north-east corner using only right and up steps.

Theorem \[\text{Schiffler, Çanakçi}\]

$\left|\left| L(G_{r/s}) \right|\right| = r$ and $\left|\left| L(\hat{G}_{r/s}) \right|\right| = s$

The notation $\hat{G}_{r/s}$ means the snake graph from the continued fraction $[a_2, a_3, \ldots, a_n]$. 

Example: The 7 lattice paths in $G_{7/3}$ are

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Lattice Paths

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**Example:** The 7 lattice paths in $G_{7/3}$ are

```
\begin{align*}
\text{Path 1} & \quad \text{Path 2} & \quad \text{Path 3} & \quad \text{Path 4} \\
\text{Path 5} & \quad \text{Path 6} & \quad \text{Path 7}
\end{align*}
```
A Partial Order on Paths

Example:

\[ L\left( \frac{G_7}{3} \right) \]
There is a partial order on the lattice paths in $G_{r/s}$ so that locally

\[
\square < \square
\]
A Partial Order on Paths

There is a partial order on the lattice paths in $G_{r/s}$ so that locally

| | < | | |

**Example:** $L(G_{7/3})$
What Do $q$-Rationals Count?

Define the height or rank of a lattice path as how many steps it takes to get to it from the minimal path.

Theorem

1. The coefficient of $q^k$ in $R(q)$ is the number of lattice paths in $G_r$ of height $k$.
2. The coefficient of $q^k$ in $S(q)$ is the number of lattice paths in $\hat{G}_r$ of height $k$.

REU Exercise 9.3: Write down all the lattice paths in $G_8$ and draw the Hasse diagram. (It should agree with Exercise 9.2(d)!)
What Do $q$-Rationals Count?

Define the *height* or *rank* of a lattice path as how many steps it takes to get to it from the minimal path.
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Let \( \left[ \frac{r}{s} \right]_q = \frac{R(q)}{S(q)} \). Then:

1. The coefficient of \( q^k \) in \( R(q) \) is the number of lattice paths in \( G_{r/s} \) of height \( k \).
2. The coefficient of \( q^k \) in \( S(q) \) is the number of lattice paths in \( \hat{G}_{r/s} \) of height \( k \).
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**REU Exercise 9.3:** Write down all the lattice paths in \( G_{8/5} \) and draw the Hasse diagram.

(It should agree with Exercise 9.2(d)!)
What Else Do They Count?

Some Suggested Presentations:

- **T-paths:** Schiffler, R. “A cluster expansion formula (A case)”. The Electronic Journal of Combinatorics 15.1 (2008) R64

- **Perfect matchings:** Çanakçi, I., Schiffler, R., “Cluster algebras and continued fractions”. Compositio Mathematica 154.3 (2018): 565-593


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- **$T$-paths:**

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- **T-paths:**

- **Perfect matchings:**

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- **All of the above:**
1 q-Analogues

2 Definition

3 How to Compute

4 What Does it Count?

5 Cluster Algebras

6 q-Real Numbers
Consider an $n$-gon. Choose a triangulation. Label the edges $e_1, ..., e_n$. Label the diagonals $x_1, ..., x_{n-3}$. The cluster algebra is a subring of $\mathbb{Q}(x_1, ..., x_{n-3}, e_1, ..., e_n)$. The boundary labels $e_1, ..., e_n$ are the "frozen" variables.
Consider an $n$-gon
The Cluster Algebra of a Polygon

- Consider an \(n\)-gon
- Choose a triangulation
Consider an $n$-gon

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Label the edges $e_1, \ldots, e_n$
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The quiver is ...
Mutations are “Flips”

\[ x_1' = \frac{e_1 e_4 + e_5 x_2}{x_1} \]
Snake Graphs

Construct the snake graph from a triangulation as we described before.

\[ e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \quad e_6 \quad e_7 \]

\[ x_1 \quad x_2 \quad x_3 \quad x_4 \]

Each lattice path \( p \) corresponds to a monomial, called the weight of the path, denoted \( \text{wt}(p) \).

**Theorem**

The cluster variable of the "longest edge" (crossing all diagonals) is

\[ x_1 \cdot x_2 \cdot \ldots \cdot x_n \]

\[ \sum_{p} \text{wt}(p) \]

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$$\frac{1}{x_1 x_2 \cdots x_n} \sum_{p} \text{wt}(p)$$
Term Count

Corollary

The cluster variable of the longest edge in $T_r/s$ has exactly $r$ terms.

REU Exercise 8.3:

(a) Compute the Laurent polynomial expression for this cluster variable using the formula in the theorem on the previous slide.

(b) Compute the same expression using a sequence of mutations (you should get the same answer!).
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The $F$-Polynomial

Label the faces of the snake graph by $y_1, \ldots, y_n$, and label the lattice paths by monomials in the $y$'s:

\[ F = 1 + y_1 + y_4 + y_1 y_4 + y_3 y_4 + y_1 y_3 y_4 + y_1 y_2 y_3 y_4 \]
The $F$-Polynomial

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Consider the cluster variable of the "longest edge" in $T_{r/s}$.

(a) $R(q) = F(q, q, \ldots, q)$

(b) The coefficient of $q^k$ in the numerator of $[r/s]q$ counts the number of terms of degree $k$ in the $F$-polynomial of the corresponding cluster variable.
Relation with $q$-Rationals

**Theorem**

Consider the cluster variable of the “longest edge” in $T_{r/s}$. 

(a) $R(q) = \prod_{i=1}^{n} \left(1 - q^{-i}\right)$

(b) The coefficient of $q^k$ in the numerator of $[r/s]q$ counts the number of terms of degree $k$ in the $F$-polynomial of the corresponding cluster variable.
Theorem

Consider the cluster variable of the “longest edge” in $T_{r/s}$.

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(b) The coefficient of $q^k$ in the numerator of $\left[ \frac{r}{s} \right]_q$ counts the number of terms of degree $k$ in the $F$-polynomial of the corresponding cluster variable.
Some REU Problems

REU Problem 9.0: (Tie-in with Gregg's Problem) Is there a combinatorial description of the $L$'s from Gregg's talk related to the $q$-rationals?

REU Problem 9.1: ("Unimodality") It is conjectured that the numerators (and denominators) of the $q$-rationals are unimodal. Any progress towards proving this would be nice, even for some non-trivial class of specific examples. In light of Chris’ talk, you could also try to prove log-concavity.
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In light of Chris’ talk, you could also try to prove log-concavity.
Outline

1. q-Analogues
2. Definition
3. How to Compute
4. What Does it Count?
5. Cluster Algebras
6. q-Real Numbers
Infinite Continued Fractions

There is a notion of infinite continued fractions. For an infinite sequence \(a_1, a_2, a_3, \ldots\), define a sequence of rational numbers (called convergents):

\[ x_n := [a_1, a_2, \ldots, a_n] \]

Then the sequence \(x_n\) converges to a real number, denoted by the infinite continued fraction \([a_1, a_2, \ldots]\).

Fact: Infinite continued fractions with coefficients that eventually repeat are exactly the quadratic irrationals.

Examples:
- \(\sqrt{2} = [1, 2, 2, 2, \ldots]\)
- \(\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, \ldots]\)
- \(\phi = \frac{1}{2}(1 + \sqrt{5}) = [1, 1, 1, 1, \ldots]\)
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\textbf{Examples:}

- $\sqrt{2} = [1, 2, 2, 2, \ldots]$
- $\sqrt{3} = [1, 1, 2, 1, 2, 1, 2, \ldots]$
- $\varphi = \frac{1}{2} (1 + \sqrt{5}) = [1, 1, 1, 1, \ldots]$
Convergence Property

The first few convergents of \( \sqrt{2} = [1, 2, 2, 2, \ldots] \) are

\[
\begin{align*}
3 & \quad 7 & \quad 17 & \quad 41 \\
2' & \quad 5' & \quad 12' & \quad 29' \\
\end{align*}
\]

The coefficients eventually "stabilize". The terms in blue remain the same in all later terms in the sequence.
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The $q$-versions are

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\begin{align*}
\left[ \frac{3}{2} \right]_q &= 1 + q^2 - q^3 + q^4 - q^5 + \cdots \\
\left[ \frac{7}{5} \right]_q &= 1 + q^3 - 2q^5 + q^6 + 3q^7 + \cdots \\
\left[ \frac{17}{12} \right]_q &= 1 + q^3 - 2q^5 + 2q^6 - q^8 + \cdots \\
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Another REU Problem

REU Problem 9.3: (Very open-ended) Almost nothing is known about the coefficients of these power series for $q$-real numbers, except for a few select specific examples computed in the original paper. Is there a pattern to these coefficients that can be predicted? Is there a combinatorial interpretation? Is it related to cluster algebras and snake graphs (see below)?

More Further Reading:
Section 7 of the paper "Cluster Algebras and Continued Fractions" (mentioned earlier)
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**More Further Reading:**
- Section 7 of the paper “Cluster Algebras and Continued Fractions” (mentioned earlier)
Here are some more papers that could be used for a presentation in weeks 3 and 4: