

Filtering Grassmannian Cohomology via k -Schur Functions

Gandhar Kulkarni¹ Ajmain Yamin² Matthew Fritz Yu³

¹BITS Pilani ²Stony Brook University ³University of California, Berkeley

Abstract

This project concerns the cohomology rings of complex Grassmannians. In 2003, Reiner and Tudose conjectured the form of the Hilbert series for certain subalgebras of these cohomology rings. We build on their work in two ways. First, we conjecture two natural bases for these subalgebras that would imply their conjecture using notions from the theory of k -Schur functions. Second, we formulate an analogous conjecture for Lagrangian Grassmannians.

Preliminaries

- The cohomology ring of the complex Grassmannian $Gr(\ell, \mathbb{C}^{\ell+k})$ with coefficients in \mathbb{Q} can be interpreted as the graded vector space:

$$R^{\ell,k} \cong \mathbb{Q}[h_1, h_2, \dots, h_k] / (e_{\ell+1}, \dots, e_{\ell+k}) = \Lambda^{(k)} / (e_{\ell+1}, \dots, e_{\ell+k})$$

where $deg(e_i) = deg(h_i) = i$, and the e_i 's are the i 'th **Jacobi-Trudi determinants**

$$\det \begin{pmatrix} h_1 & h_2 & \dots & 0 & \dots & 0 \\ 1 & h_1 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & h_1 & h_2 & \dots \\ 0 & \dots & 0 & 1 & h_1 & \dots \end{pmatrix}.$$

- $R^{\ell,k,m}$ is the subalgebra of $R^{\ell,k}$ generated by h_1, \dots, h_m . See that $\mathbb{Q} = R^{\ell,k,0} \subset R^{\ell,k,1} \subset R^{\ell,k,2} \subset \dots \subset R^{\ell,k,m} \subset \dots \subset R^{\ell,k}$
- A **partition** is a weakly decreasing sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n)$. It can be represented by its Ferrer's diagram:

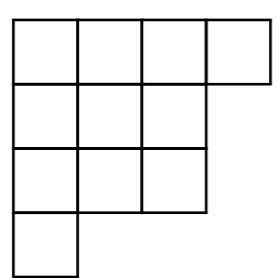


Figure 1: The Ferrer's diagram of the partition $\lambda = (4, 3, 3, 1)$.

The q -binomial coefficient is the polynomial $\begin{bmatrix} k + \ell \\ k \end{bmatrix}_q = \sum_{\lambda \in \mathcal{C}(k, \ell)} q^{|\lambda|}$.

- Given any graded vector space $R = \bigoplus_{d=0}^{\infty} R_d$ over \mathbb{Q} , the **Hilbert series** of R is $\text{Hilb}(R, q) = \sum_{d=0}^{\infty} \dim_{\mathbb{Q}}(R_d) q^d$.

The Problem (R-T Conjecture)[5]

For each $m = 0, 1, 2, \dots, \min(k, \ell)$, one has

$$\text{Hilb}(R^{\ell,k,m}, q) = 1 + \sum_{i=1}^m q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q \quad (1)$$

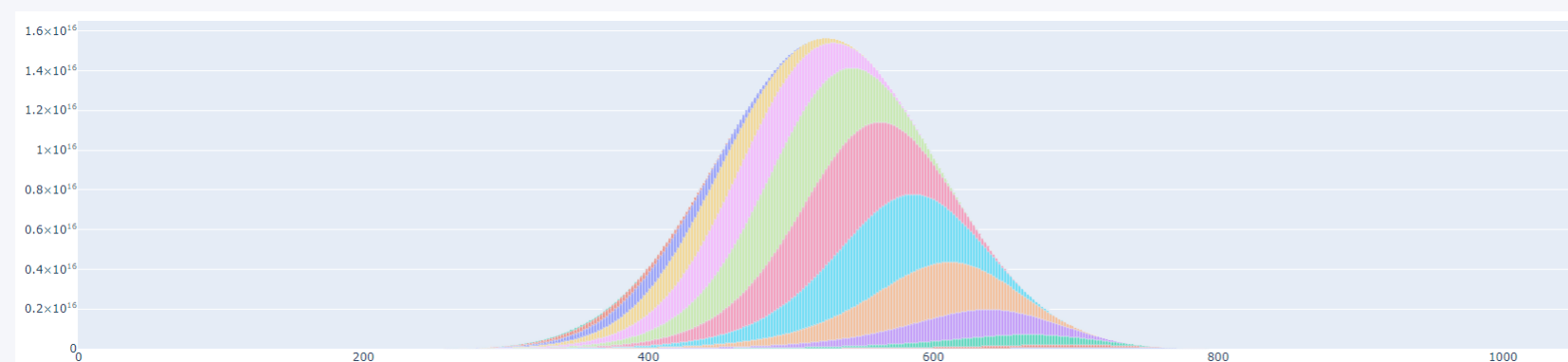


Figure 2: An illustration of the R-T Conjecture for $k = 30, l = 35$.

Boundary Cases

- One can check that for $m = 1$, this conjecture reduces to $\text{Hilb}(R^{\ell,k,1}, q) = 1 + q + \dots + q^{k\ell}$, which can be deduced from either Schubert calculus or the hard Lefschetz theorem.
- For $m = \min(k, \ell)$, this conjecture must be consistent with $\text{Hilb}(R^{\ell,k}, q) = \begin{bmatrix} \ell+k \\ \ell \end{bmatrix}_q$. We can verify that the RHS of the R-T Conjecture reduces to this q -binomial coefficient via a combinatorial interpretation of the R-T conjecture involving the notion of i -vacant partitions.

i -vacant Partitions

A k -bounded partition λ is i -**vacant** if i is the largest integer for which the complementary skew diagram $(k^{\ell(\lambda)})/\lambda$ contains an $i \times (i-1)$ rectangle in its southeast corner. We will call $(k^{\ell(\lambda)})$ the **ambient k -rectangle** of λ .

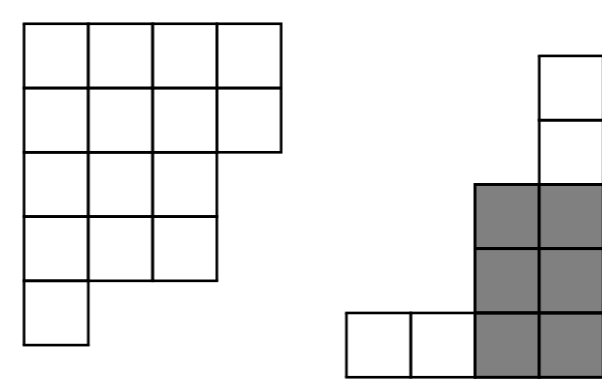
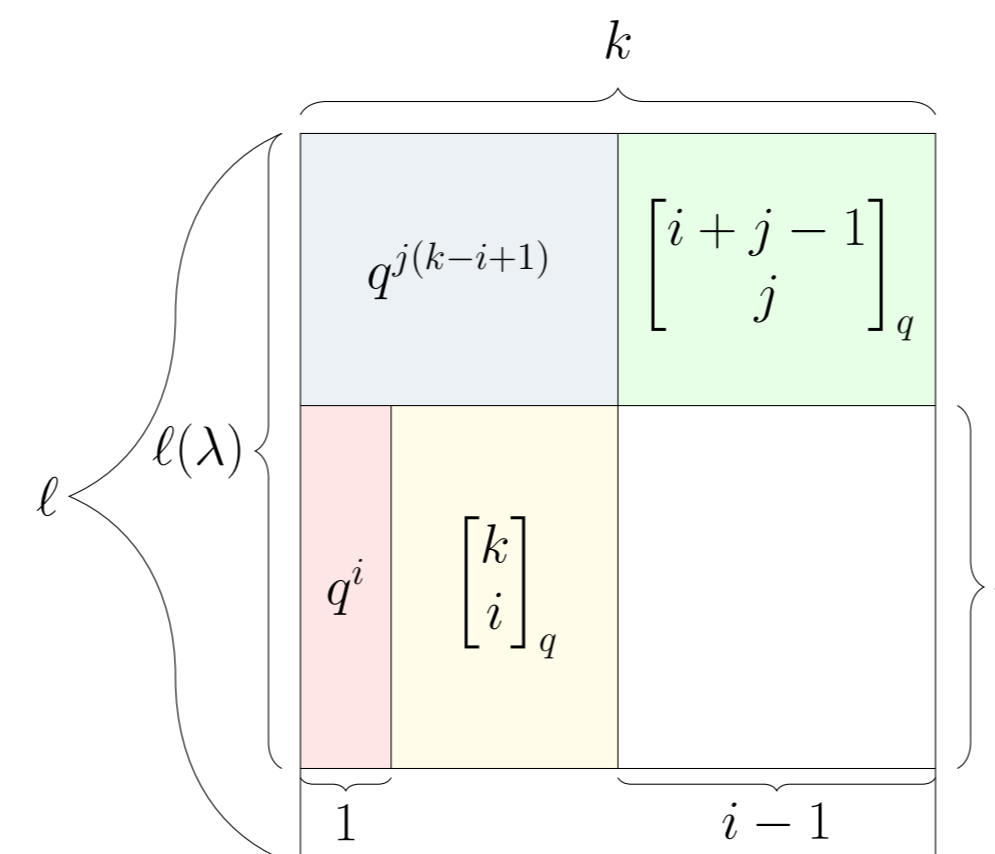


Figure 3: The 5-bounded partition $\lambda = (4, 4, 3, 3, 1)$ is 3-vacant.

Combinatorial Interpretation of the R-T Conjecture

For each $m = 0, 1, 2, \dots, \min(k, \ell)$,

$$\sum_{\substack{i\text{-vacant, } i \leq m \\ \lambda \in \mathcal{C}(k, \ell)}} q^{|\lambda|} = 1 + \sum_{i=1}^m q^i \begin{bmatrix} k \\ i \end{bmatrix}_q \sum_{j=0}^{\ell-i} q^{j(k-i+1)} \begin{bmatrix} i+j-1 \\ j \end{bmatrix}_q$$



k -conjugation

A k -**bounded partition** is a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_d)$ where $\lambda_1 \leq k$. We denote the set of all k -bounded partitions by \mathcal{P}^k .

A $(k+1)$ -**core** is a partition $\lambda = (\lambda_1 \geq \dots \geq \lambda_d)$ where no box has hook-length equal to $k+1$. We denote the set of all $(k+1)$ -cores by \mathcal{C}^{k+1} .

There is a bijection between \mathcal{P}^k and \mathcal{C}^{k+1} :

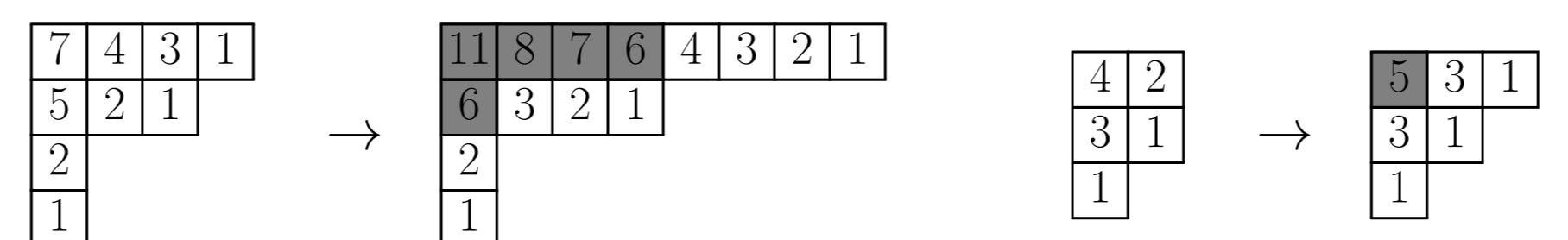


Figure 4: $(4, 3, 1, 1)$ is 4-bounded and a 6-core.

Theorem: (Lapointe, Lascoux and Morse, 2003) [3] The map $\omega(k) : \mathcal{P}^k \rightarrow \mathcal{P}^k$ defined by sending a k -bounded partition λ to $\lambda^{\omega(k)} := \mathbf{p}(\mathbf{c}(\lambda)')$ is an involution, where $(-)'$ denotes usual conjugation. We call $\lambda^{\omega(k)}$ the k -conjugate of λ .

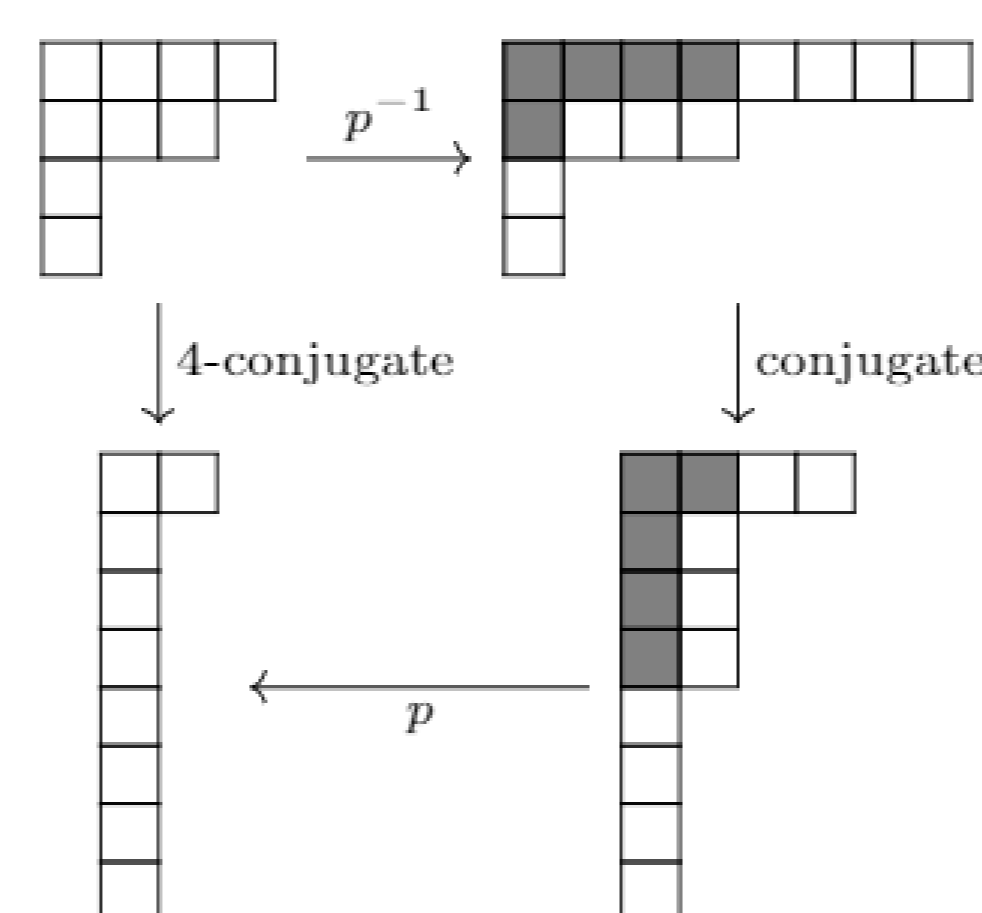
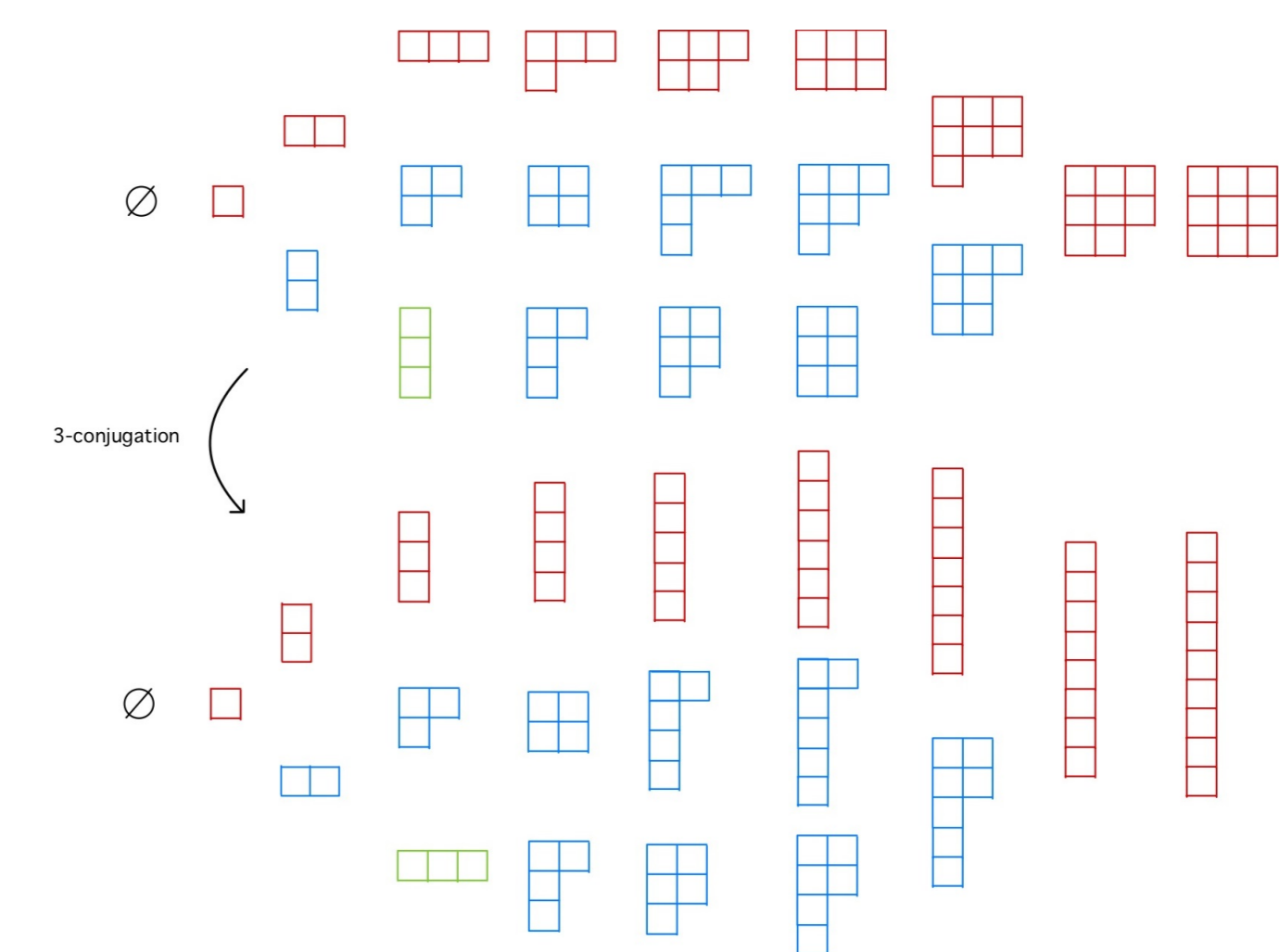


Figure 5: An illustration of the R-T Conjecture for $k = 30, l = 35$.

Conjectured Filtered Bases

Proposition: For any $i = 1, 2, \dots, \min(k, \ell)$, one has

$$\sum_{\substack{i\text{-vacant} \\ \lambda \in \mathcal{C}(k, \ell)}} q^{|\lambda|} = \sum_{\substack{\mu: \mu_1 = i, \\ \mu^{\omega(k)} \in \mathcal{C}(k, \ell)}} q^{|\mu|}.$$



Conjecture:

- (a) The set $\{h_\lambda \mid \lambda \in \mathcal{P}^{k, \ell}\}$ is a basis of $R^{\ell, k}$; moreover,
- (b) the set $\{h_\lambda \mid \lambda \in \mathcal{P}^{k, \ell, m}\}$ is a basis of $R^{\ell, k, m}$ for all $m = 1, \dots, \min(k, \ell)$

- (a) will show half of the R-T conjecture (LHS \geq RHS)
- (b) implies the full R-T conjecture

Lagrangian Analogue

For a strictly decreasing partition $\lambda = (\lambda_1 > \dots > \lambda_\ell)$, we define its **shifted Young diagram** to be a diagram with λ_i boxes in row i with each row shifted one unit right of the previous one. An **ambient triangle of size n** , denoted as Δ_n , is a shifted Young diagram $\lambda = (n > n-1 > \dots > 1)$.

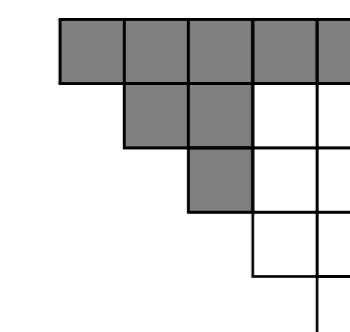


Figure 6: An ambient triangle Δ_5 and a shifted Young diagram $\lambda = (5, 2, 1)$ colored gray.

In Lie type C, we replace $Gr(\ell, \mathbb{C}^{k+\ell})$ by the Lagrangian Grassmannian $\text{LG}(n, \mathbb{C}^{2n})$ and define the ring $R_{\text{LG}}^n := H^*(\text{LG}(n, 2n); \mathbb{Q})$. It is known that

$$R_{\text{LG}}^n \cong [e_1, e_2, \dots, e_n] / \left(e_i^2 + 2 \sum_{k=1}^{n-i} (-1)^k e_{i+k} e_{i-k} \right)_{i=1, 2, \dots, n}$$

$$\text{Hilb}(R^n, q) = \sum_{\lambda \in \Delta^n} q^{|\lambda|} = (1+q)(1+q^2)(1+q^3) \dots (1+q^n).$$

The R-T Conjecture (Type C Analogue)

For each $m = 0, 1, \dots, n$,

$$\text{Hilb}(R^{n,m}, q) = 1 + \sum_{\substack{1 \leq i \leq m \\ i \text{ odd}}} q^i \sum_{j=0}^{n-i} q^{\binom{j+1}{2}} \begin{bmatrix} i+j \\ i \end{bmatrix}_q. \quad (2)$$

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