Filtering Grassmannian Cohomology via $k$-Schur Functions

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Abstract

This project concerns the cohomology rings of complex Grassmannians. In 2003, Reiner and Tudose conjectured the form of the Hilbert series for certain subalgebras of these cohomology rings. We build on their work in two ways. First, we conjecture two natural bases for these subalgebras that would imply their conjecture using notions from the theory of $k$-Schur functions. Second, we formulate an analogus conjecture for Lagrangian Grassmannians.

Preliminaries

• The cohomology ring of the complex Grassmannian $Gr(k, C^{n+k})$ with coefficients in $Q$ can be interpreted as the graded vector space $R(k) = \bigoplus_{n,k} R_n(k)$, where $R_n(k)$ is the subalgebra of $R(k)$ generated by $h_{1,\ldots,k}$, . . . , $h_{n,k}$. Given any graded vector space $R$, $R(k)$ is the subalgebra of $R(k)$ generated by $h_{1,\ldots,k}$, . . . , $h_{n,k}$.

• $R(k,m)$ is the subalgebra of $R(k)$ generated by $h_{1,\ldots,m}$. See that $R(k) = R(k,0) \subset R(k,1) \subset \cdots \subset R(k,m) \subset \cdots \subset R(k)$. $R(k,m)$ is the algebra of $k$-Schur functions.

• For $m = \min((k, l), (k, m))$, this conjecture must be consistent with $H(k^2, (4,1)) = 1 + q + q^2 + q^3$, which can be deduced from Schubert calculus or the hard Lefschetz theorem.

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• For a strictly decreasing partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$, there is a bijection between $P(k)$ and $\ell$-core partitions.

The R-T Conjecture

A $k$-bounded partition $\lambda$ is $\ell$-vaccum if $\ell \subset \lambda$, where $\ell$ is an integer sequence.

Combinatorial Interpretation of the R-T Conjecture

For each $\nu = 0, 1, 2, \ldots, \min(\lambda, k)$, we have:

$$\binom{\lambda}{\nu} = 1 + \sum_{i=1}^{\nu} \binom{\lambda}{i} \sum_{\nu' \geq i} \binom{\lambda'}{\nu'-i} \binom{\nu-i}{j}$$

References