DIHEDRAL SIEVING PHENOMENA

SUJIT RAO AND JOE SUK

Abstract. Cyclic sieving is a well-known phenomenon where certain interesting polynomials, especially q-analogues, have useful interpretations related to actions and representations of the cyclic group. We define sieving for an arbitrary group G and study it for the dihedral group $I_2(n)$ of order $2n$. This requires understanding the generators of the representation ring of the dihedral group. For n odd, we exhibit several instances of "dihedral sieving" which involve the generalized Fibonomial coefficients, recently studied by Sagan *et al.*. In addition, we give some recursively-defined polynomials which may help prove instances of dihedral sieving for even n.

1. Introduction

The cyclic sieving phenomenon was originally studied by Reiner, Stanton, and White in [11] in 2004 and has, since then, led to a greater understanding of the combinatorics of various finite sets with a natural cyclic action. In particular, cyclic sieving allows one to count the fixed points of a cyclic action on a finite set through an associated generating function. These generating functions often appear in other contexts, such as the generating function for permutation statistics related to Coxeter groups and as the Hilbert series of some interesting graded ring. Proofs of cyclic sieving also tend to have interesting connections with representation theory.

We start by precisely defining cyclic sieving:

Definition 1.1 (cyclic sieving phenomenon)**.** Let X be a finite set, $X(q)$ be a polynomial with nonnegative integral coefficients, and $\mathbb{Z}/n\mathbb{Z}$ be a cyclic group of order n with a group action on X. Let $\omega_n : \mathbb{Z}/n\mathbb{Z} \hookrightarrow \mathbb{C}^\times$ be the map defined by $m \mapsto e^{2\pi m i/n}$. Then, we say the triple $(X, X(q), \mathbb{Z}/n\mathbb{Z})$ exhibits the **cyclic sieving phenomenon** if for all $c \in \mathbb{Z}/n\mathbb{Z}$,

$$|X(q)|_{q=\omega(c)} = |\{x \in X : c(x) = x\}| \tag{1.2}$$

As $X(1) = |X|$, $X(q)$ is both a q-analogue of the cardinality and a generating function for the number of $\mathbb{Z}/n\mathbb{Z}$-orbits of size k in X. Before discussing some classic examples of cyclic sieving, we recall the definition of the q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$. First, let $[n]_q = 1 + q + q^2 + \cdots + q^{n-1}$ and let $[n]!_q = [n]_q[n-1]_q \cdots [2]_q[1]_q$. Then, the q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q[n-k]!_q} \tag{1.3}$$

This is a rational function in q. It is not immediately obvious, but the q-binomial coefficient can also be shown to be a polynomial in q with nonnegative integral coefficients. MacMahon's q-Catalan number, defined similarly,

$$C_n(q) := \frac{1}{[n+1]_q}\begin{bmatrix} 2n \\ n \end{bmatrix}_q \tag{1.4}$$

is also a polynomial in q with nonnegative integral coefficients.

Now, we discuss some examples of cyclic sieving given in the seminal paper in [11]. All of these are cases where the natural cyclic action of C on the set $[n] := \{1, \ldots, n\}$ induces an action on some collection of subsets of $[n]$. Sometimes the action $C \circlearrowright [n]$ is interpreted geometrically as rotations of an n-gon.

Example 1.5. Let C be a cyclic group of order n. Then, the following triples $(X, X(q), C)$ exhibit cyclic sieving phenomenon.

(1) Let $X = \{\text{size } k \text{ multisubsets of } [n]\}$ and let $X(q) = n + k - 1k_q$.
(2) Let $X = \{\text{size } k \text{ subsets of } [n]\}$ and let $X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q$.
(3) Let $X = \{\text{dissections of a convex } n\text{-gon using } k \text{ diagonals}\}$ and let $X(q) = \frac{1}{[n+k]_q}\begin{bmatrix} n+k \\ k+1 \end{bmatrix}_q n - 3k_q$.
(4) Let $X = \{\text{triangulations of a regular } n\text{-gon}\}$ and let $X(q) = C_{n-2}(q)$.
(5) Let $X = \{\text{noncrossing partitions of an } n\text{-gon}\}$ and let $X(q) = C_n(q)$.
(6) Let $X = \{\text{noncrossing partitions of an } n\text{-gon using } n-k \text{ parts}\}$ and let $X(q) = \frac{1}{[n]_q}\begin{bmatrix} n \\ k \end{bmatrix}_q \begin{bmatrix} n \\ k+1 \end{bmatrix}_q q^{k(k+1)}$.

Note that in case (3), $X(q)$ is a q-analogue of $f(n,k) = \frac{1}{n+k} n + kk + 1n - 3k$, a formula for the number of dissections of a convex n-gon using k diagonals and in case (6), $X(q)$ is a q-analogue of $N(n,k) = \frac{1}{n} nknk + 1$, the Narayana number which counts the number of non-crossing partitions of $[n]$ using $n-k$ parts. Both of these are also polynomials in q for virtually the same reasons their corresponding numbers are integers. In general, if a formula involving binomial coefficients satisfies some nice divisibility properties, the corresponding q-analogue will also satisfy the analogous properties. Also note that case (4) is a specific example of case (3) where $k = n-3$.

Every cyclic action can be equipped with a generating polynomial $X(q)$ to obtain an instance of cyclic sieving. If we require $X(q)$ be of degree at most $n-1$, then the choice of polynomial is unique. Thus, cyclic sieving can be made ubiquitous. However, the interest and fascination of this subject stem from the fact that some of these associated generating polynomials will be interesting. Most cyclic actions will have uninteresting generating polynomials, but cases, such as the q-binomials or the q-Catalan numbers used above, appear in many other mathematical contexts.

Typically, we have an educated guess for an appropriate cyclic sieving polynomial for a particular cyclic action which has already been used in another context. Proving that a chosen interesting polynomial, or the cyclic action equipped with that polynomial, is an instance of cyclic sieving often happens in two ways: via direct computation of the generating polynomial or through an understanding of the representation-theoretic perspective of cyclic sieving phenomena. We provide two helpful results for taking the latter route from [11]. Let $\mathbb{C}^X$ be the permutation representation of the action $\mathbb{Z}/n\mathbb{Z} \circlearrowright X$ (i.e. the finite-dimensional complex vector space with basis indexed by elements of X).

Theorem 1.6. [Proposition 2.1 of [11]] Consider a triple $(X, X(q), C)$ as in the setup of 1.1. Let A_X be a graded $\mathbb{C}$-vector space $A_X = \oplus_{i\geqslant 0} A_{X,i}$ having $\sum_{i\geqslant 0} \dim_{\mathbb{C}} A_{X,i} q^i = X(q)$. A_X can be considered a representation of C in which each $c \in C$ acts on the graded component $A_{X,i}$ by the scalar $\omega(c)^i$. Then, $(X, X(q), C)$ has cyclic sieving if and only if we have an isomorphism of C-representations $A_X \cong \mathbb{C}^X$.

As discussed before, the proposition above implies the choice of generating polynomial $X(q)$ is unique if we require it is of degree $\leqslant n-1$. For a rational representation $\rho : GL_n(\mathbb{C}) \to GL_n(V)$, let $\chi_\rho(x_1, \ldots, x_N)$ be the trace on V of any diagonalizable element of $GL_N(\mathbb{C})$ having eigenvalues $x_1, \ldots, x_N$.

Theorem 1.7. [Lemma 2.4 of [11]] Let $\rho : GL_n(\mathbb{C}) \to GL_n(V)$ be a highest-weight representation. Assume V has a basis $\{v_x\}_{x\in X}$ which is permuted by $\mathbb{Z}/n\mathbb{Z}$ in the following way:

$$c(v_x) = v_{c(x)} \text{ for all } c \in \mathbb{Z}/n\mathbb{Z}, x \in X$$

Then, let $X(q)$ be the (twisted) principal specialization

$$X(q) = \chi_\rho(1, q, \ldots, q^{N-1})$$

Then, $(X, X(q), C)$ has cyclic sieving.

In particular, the above lemma can be used to prove cyclic sieving for $\mathbb{Z}/n\mathbb{Z} \circlearrowright X$ where $X = \{k\text{-multi-subsets of } [n]\}$. More precisely, we take $V = V^\lambda$, the irreducible representation of $GL_N(\mathbb{C})$ with highest weight $\lambda = (k) \vdash k$ (the partition of k with one part), and the specialization of the character value becomes the q-analogue of the Weyl character formula or the hook-length formula

$$\chi_\rho(1, q, \ldots, q^{N-1}) = s_\lambda(1, q, \ldots, q^{N-1}) = q^{b(\lambda)} \frac{[n]!_q}{\prod_{\text{cells } x \in \lambda} [h(x)]_q} = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$$

where $h(x)$ is the hook-length of λ at x.

Defining sieving phenomena for other finite groups is a natural route of generalization for understanding other group actions. For the case of abelian groups, or direct products of cyclic groups, a natural definition of polycyclic sieving using a multivariate generating function was defined with accompanying instances, or examples, in [3].

In this paper, we give a new definition of sieving phenomenon for finite groups motivated by the representation theoretic perspective of cyclic sieving provided above and apply this to the case of dihedral group actions. We prove that the natural dihedral action in situations (1), (2), (5), and (6) in 1.5 have dihedral sieving for n odd. The analogous generating polynomials we obtain are all defined in terms of generalized Fibonacci polynomials, first studied by Hoggatt and Long in [9], and their induced generalized Fibonomial coefficients, recently studied by Sagan *et al.* in [2]. For n even, we provide some examples of recursively defined polynomials which "almost" give a generating polynomial for dihedral sieving and discuss the fundamental differences with the odd n case. We also discuss the difficulties of proving dihedral sieving for the dihedral group actions in situations (3) and (4) of 1.5 and other instances of cyclic sieving which may yield interesting generating polynomials when moving to the dihedral setting.

2. Preliminaries

Throughout the paper, we will use the following notation

Notation 2.1. Let $C = \langle c \rangle$ be a cyclic group of order n. Let $\omega : C \to \mathbb{C}^\times$ be an embedding of C defined by $c \mapsto e^{2\pi i/n}$. This can also be considered a 1-dimensional complex representation of C.

If V is a representation of a group G, we will use χ_V to refer to its character. If $x \in G$, $\chi_V(x)$ is the value of the character at x. If $C \subset G$ is a conjugacy class, then $\chi_V(C)$ is the value of the character on C as a class function. In the case where $G = GL_N(\mathbb{C})$, we use $\chi_V(\mathrm{diag}(x_1, \ldots, x_N))$ to denote the value of the character on any diagonalizable element of $GL_N(\mathbb{C})$ having eigenvalues $x_1, \ldots, x_N$.

We will first give an equivalent definition of cyclic sieving based on representation theory, which is more suitable for adapting to other groups.

Definition 2.2. Let G be a group and R be a commutative unital ring. Let A be the set of isomorphism classes of finite-dimensional G-representations over $\mathbb{C}$. The **representation ring** of G with coefficients in R is

$$\mathrm{Rep}(G; R) = R[A]/(I + J)$$

where $R[A]$ is the polynomial ring over R generated by A, and I and J are the ideals

$$I = (\{[U \oplus V] - ([U] + [V])\})$$
$$J = (\{[U \otimes V] - [U][V]\})$$

and $[U]$ denotes the isomorphism class of a G-representation U.

Remark 2.3. The above definition is the same as saying that $\mathrm{Rep}(G; R)$ is the Grothendieck ring of the symmetric monoidal category of G-representations over $\mathbb{C}$.

We will often use the following facts about the representation ring of a group.

Fact 2.4. *The representation ring* $\mathrm{Rep}(G; R)$ *is a free R-module with a basis given by isomorphism classes of irreducible representations.*

Fact 2.5. *The map defined by*

$$\begin{aligned} \mathrm{Rep}(G; \mathbb{C}) &\longrightarrow \mathrm{ClFun}(G) \\ [V] &\longmapsto \chi_V \end{aligned}$$

which sends an isomorphism class of a representation to its character, is an isomorphism of rings.

We can now state an equivalent definition of cyclic sieving based on the representation ring.

Theorem 2.6. [Proposition 2.1 of [11]] Let $(X, X(q), C)$ be a triple where C acts on X and $X(q) \in \mathbb{N}[q]$. This triple has cyclic sieving if and only if $\mathbb{C}^X = X(\omega)$ in $\mathrm{Rep}(C; \mathbb{Z})$.

This motivates the following, more general definition.

Definition 2.7. Let G be a group and $\rho_1, \ldots, \rho_k$ be representations of G over $\mathbb{C}$. Let X be a G-set and $X(q_1, \ldots, q_k) \in R[q_1, \ldots, q_k]$ for some ring $R \subseteq \mathbb{C}$. Then the quadruple $(X, X(q_1, \ldots, q_k), (\rho_1, \ldots, \rho_k), G)$ has **G-sieving** if $\mathbb{C}^X = X(\rho_1, \ldots, \rho_k)$ in $\mathrm{Rep}(G; \mathbb{C})$.

Example 2.8. Let $G = C$ and $\rho_1 = \omega$. Then the definition G-sieving above agrees with the usual definition of cyclic sieving.

Example 2.9. Let $G = C_n \times C_m$ be a product of cyclic groups. Let $\rho_1 = \omega_n \otimes 1_m$, where $\omega_n : C_n \to \mathbb{C}^\times$ is an embedding and 1_m is the trivial representation of C_m, and similarly let $\rho_2 = 1_n \otimes \omega_m$. Then the definition of G-sieving above agrees with the definition of bicyclic sieving given in [3].

We are typically interested in cases where the polynomial X can be written in an interesting way, such as product formulas based on q-analogues.

3. Dihedral Sieving

By the observations in the previous section, we can describe $I_2(n)$-sieving in terms of a generating set of $\mathrm{Rep}(I_2(n); \mathbb{C})$. We first need a description of the representation ring of the dihedral group. We start by briefly recalling the irreducible representations of $I_2(n)$. We adhere to the presentation

$$I_2(n) = \langle r, s | r^n = s^2 = e, rs = sr^{-1} \rangle \tag{3.1}$$

The irreducible representations and, thus, the representation ring, will depend on whether n is odd or even. For n odd, there are two 1-dimensional irreducible representations, the trivial representation $\mathbb{1}$ and the determinant representation, and $\lfloor n/2 \rfloor$ 2-dimensional irreducible representations: the representations z_m which sends r to a counterclockwise rotation matrix of $2\pi m/n$ radians and s to a reflection matrix where $m \in [1, n/2) \cap \mathbb{N}$. For n even, there are four 1-dimensional irreducible representations: $\mathbb{1}$, det, χ_b which sends $\langle r^2, s\rangle$ to 1 and r to -1, and $\det \cdot \chi_b$. There are $n/2-1$ 2-dimensional irreducible representations, defined the same way as the n odd case.

We refer to [6] for the following. First, we have (for both n odd and even) the following relations among the irreducible representations:

$$\begin{aligned}\det{}^2 &= 1\\ \det \cdot z_k &= z_k\\ z_{k+1} &= z_k z_1 - z_{k-1} \text{ if } k \leqslant \frac{n-3}{2} \text{ and where } z_0 = 1 + \det\end{aligned}$$

Thus, $\mathrm{Rep}(I_2(n); \mathbb{C})$ is generated by $\det, z_1$ for n odd and by $\det, z_1, \chi_b$ for n even.

The examples of dihedral sieving we exhibit are all for odd n and make use of the generalized Fibonacci polynomials and Fibonomial coefficients defined in [2]. We state some of their results here.

Definition 3.2 (Equation (2) of [2])**.** The **generalized Fibonacci polynomials** are a sequence $\{n\}_{s,t}$ of polynomials in $\mathbb{N}[s,t]$ defined inductively by

$$\begin{aligned}\{0\}_{s,t} &= 0\\ \{1\}_{s,t} &= 1\\ \{n+2\}_{s,t} &= s\{n+1\}_{s,t} + t\{n\}_{s,t}.\end{aligned}$$

We also define

$$\{n\}!_{s,t} = \{n\}_{s,t}\{n-1\}_{s,t}\cdots\{1\}_{s,t}$$

and the **Fibonomial coefficient** to be

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{s,t} = \frac{\{n\}!_{s,t}}{\{k\}!_{s,t}\{n-k\}!_{s,t}}.$$

Proposition 3.3 (Equation (7) of [2])**.** *The Fibonomial coefficient* $\left\{\begin{smallmatrix} n \\ k \end{smallmatrix}\right\}_{s,t}$ *is a polynomial in s and t with nonnegative integer coefficients.*

Proposition 3.4 (Equation (7) of [2])**.** *Let* $X = \frac{s+\sqrt{s^2+4t}}{2}$ *and* $Y = \frac{s-\sqrt{s^2+4t}}{2}$. *Then*

$$\{n\}_{s,t} = Y^{n-1}\, [n]_q\big|_{q=X/Y}$$

and

$$\begin{Bmatrix} n \\ k \end{Bmatrix}_{s,t} = Y^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q \Bigg|_{q=X/Y}.$$

4. Dihedral Action on k-subsets of $\{1, \dots, n\}$

Proposition 4.1. *Let n be odd and* $X = \binom{[n]}{k}$. *Then the quadruple*

$$\left(X, \begin{Bmatrix} n \\ k \end{Bmatrix}_{s,t}, (z_1, -\det), I_2(n)\right)$$

exhibits dihedral sieving.

Proof. Let ξ_n be the primitive n-th root of unity $e^{2\pi i/n}$. Using the presentation in 3.1, let C be a conjugacy class of $I_2(n)$.

First, we compute the character values of $\mathbb{C}^X$, the permutation representation. Clearly,

$$\chi_{\mathbb{C}^X}(r^\ell) = \begin{bmatrix} n \\ k \end{bmatrix}_{q=\xi_n^\ell}$$

by cyclic sieving where the input r^ℓ can refer to either the group element $r^\ell \in I_2(n)$ or to the conjugacy class $\{r^\ell, r^{n-\ell}\}$ in $I_2(n)$. An easy counting argument shows the number of fixed points of a reflection s in the action is

$$\binom{\frac{n-1}{2}}{\lfloor k/2 \rfloor} = \begin{bmatrix} n \\ k \end{bmatrix}_{q=\xi_2}$$

Now, the proof is a direct computation. Consider the sequence of generalized Fibonacci polynomials $\{n\}_{s=z_1,t=-\det}$ and their induced generalized Fibonomial $\left\{{n \atop k}\right\}_{s=z_1,t=-\det}$. By 3.4,

$$\left\{{n \atop k}\right\}_{s=z_1,t=-\det} = Y^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q=X/Y}$$

where X, Y are the roots of the characteristic equation of the recurrence defined by $\{n\}_{z_1,-\det}$. Namely,

$$X = \begin{cases} \frac{z_1(C)+\sqrt{z_1(C)^2+4t}}{2} = \xi_n^\ell & C = \{r^\ell\} \\ 1 & C = \{s, sr, sr^2, \dots\} \end{cases}$$

$$Y = \begin{cases} \frac{z_1(C)-\sqrt{z_1(C)^2+4t}}{2} = \xi_n^{-\ell} & C = \{r^\ell, r^{n-\ell}\} \\ \text{-1} & C = \{s, sr, sr^2, \dots\} \end{cases}$$

$$X/Y = \begin{cases} \xi_n^{2\ell} & C = \{r^\ell, r^{n-\ell}\} \\ -1 & C = \{s, sr, sr^2, \dots\} \end{cases}$$

with

$$\gcd(n,\ell) = \gcd(n, 2\ell) \implies \begin{bmatrix} n \\ k \end{bmatrix}_{q=\xi_n^{2\ell}} = \begin{bmatrix} n \\ k \end{bmatrix}_{q=\xi_n^{\ell}}$$

so that

$$\begin{aligned} \left\{{n \atop k}\right\}_{z_1(C),-\det(C)} &= \begin{cases} (\xi_n^{-\ell})^{k(n-k)} \left[{n \atop k}\right]_{\xi_n^\ell} & C = \{r^\ell, r^{n-\ell}\} \\ (-1)^{k(n-k)} \left[{n \atop k}\right]_{\xi_2} & C = \{s, sr, sr^2, \dots\} \end{cases} \\ &= \begin{cases} (\xi_n^{-\ell})^{-k^2} \left[{n \atop k}\right]_{\xi_n^\ell} & C = \{r^\ell, r^{n-\ell}\} \\ \left[{n \atop k}\right]_{\xi_2} & C = \{s, sr, sr^2, \dots\} \end{cases} \end{aligned}$$

In fact since

$$\begin{bmatrix} n \\ k \end{bmatrix}_{\xi_n^\ell} = \begin{bmatrix} n \\ k \end{bmatrix}_{q=\xi_{n/\gcd(n,\ell)}} = \begin{cases} \binom{\gcd(n,\ell)}{\frac{k\gcd(n,\ell)}{n}} & \frac{n}{\gcd(n,\ell)} | k \\ 0 & \frac{n}{\gcd(n,\ell)} \nmid k \end{cases}$$

we have that $n | k\gcd(n,\ell) \implies n | \ell k$ meaning

$$\left\{{n \atop k}\right\}_{z_1(C),-\det(C)} = \begin{cases} \left[{n \atop k}\right]_{\xi_n^\ell} & C = \{r^\ell, r^{n-\ell}\} \\ \left[{n \atop k}\right]_{\xi_2} & C = \{s, sr^2, \dots\} \end{cases}$$

□

The other instances of dihedral sieving in this paper all involve generalized Fibonomial coefficients and can be proven using a similar computation. Note that the fact that n was odd was explicitly used in the proof above (namely that $\gcd(n,\ell) = \gcd(n, 2\ell)$).

5. Dihedral Action on k-multisubsets of $\{1, \dots, n\}$

First, we discuss a generalization of 1.7 which will help prove dihedral sieving for k-multisubsets in the same manner as the representation-theoretic proof of Theorem 1.1 in [11]. Given a highest-weight $\mathrm{GL}_n(\mathbb{C})$-representation V, we can consider its character specialization $\chi_V(\mathrm{diag}(a^{n-1}, a^{n-2}b, \dots, ab^{n-2}, b^{n-1}))$ as in 1.7 for some variables a, b. This is a symmetric polynomial in $a^{n-1}, a^{n-2}b, \dots, ab^{n-2}, b^{n-1}$ and, thus, also in a, b. This means the character specialization can be expressed as a polynomial in $a + b, ab$.

Proposition 5.1. *Let n be odd and V be a highest-weight $\mathrm{GL}_n(\mathbb{C})$-representation. Considering $I_2(n)$ as a subgroup of $\mathrm{GL}_n(\mathbb{C})$, assume that V has a basis indexed by a (finite) set X which is permuted by $I_2(n)$. Let p be the unique polynomial in two variables such that*

$$p(a+b, ab) = \chi_V(\mathrm{diag}(a^{n-1}, a^{n-2}b, \dots, ab^{n-2}, b^{n-1}))$$

noting that the right-hand side is a symmetric function in a and b. Then $(X, p, (z_1, -\det), I_2(n))$ exhibits dihedral sieving.

Proof. Let C be a conjugacy class in $I_2(n)$ and $X = \frac{s+\sqrt{s^2+4t}}{2}$, $Y = \frac{s-\sqrt{s^2+4t}}{2}$ where $s = \chi_{z_1}(C)$ and $t = \chi_{-\det}(C)$. It is straightforward to check that the eigenvalues of any element in C are $X^{n-1}, X^{n-2}Y, \dots, Y^{n-1}$, and thus

$$\begin{aligned} \chi_V(C) &= \chi_V(X^{n-1}, X^{n-2}Y, \dots, XY^{n-2}, Y^{n-1}) \\ &= p(X+Y, XY) \\ &= p(\chi_{z_1}(C), \chi_{-\det}(C)). \end{aligned}$$

Thus $V = p(z_1, -\det)$ in $\mathrm{Rep}(I_2(n);\mathbb{C})$. □

The application of 1.7 to prove cyclic sieving for k-multisubsets of $[n]$ immediately generalizes with the above proposition.

Proposition 5.2. *Let n be odd and $X = \left(\!\binom{[n]}{k}\!\right)$. Then the quadruple*

$$\left(X, \left\{ {n+k-1 \atop k} \right\}_{s,t}, (z_1, -\det), I_2(n)\right)$$

exhibits dihedral sieving.

Proof 1. Let $V = \mathrm{Sym}^k(\mathbb{C}^n) = V_\lambda$, the irreducible representation of $\mathrm{GL}_n(\mathbb{C})$ of highest weight $\lambda = (k) \vdash k$. Then, depending on the conjugacy class the character value will be

$$\chi_V(X^{n-1}, X^{n-2}Y, \ldots, Y^{n-1}) = \begin{cases} \chi_V(\omega(r^\ell)^{n-1}, \omega(r^\ell)^{n-3}, \ldots) = s_\lambda(\omega(r^\ell)^{n-1}, \omega(r^\ell)^{n-3}, \ldots) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q=\xi_n^\ell} \\ \chi_V(1, -1, \ldots) = s_\lambda(1, -1, \ldots) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q=-1} \end{cases}$$

by the usual cyclic sieving character specialization given in 1.7. □

Alternatively, dihedral sieving can be proven using a computation similar to 4.

Proof 2. Let ξ_n be the primitive n-th root of unity $e^{2\pi i/n}$. Using the presentation in 3.1, let C be a conjugacy class of $I_2(n)$.

First, we compute the character values of $\mathbb{C}^X$. Clearly, $\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q=\xi_n^\ell}$ counts the fixed points of the rotation conjugacy classes in the action by Theorem 1.1(a) of [11]. For the class $\{s, sr, sr^2, \ldots\}$, we claim $\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q=\xi_2}$ counts the fixed points. For the sake of induction, first we investigate the case of n even and the reflection conjugacy class $\{sr, sr^2, \ldots, \}$. In this case, it is easy to see that the multisubsets fixed by a reflection of this form are in bijective correspondence with those fixed by a rotation by π radians. Thus, $\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{\xi_2}$ counts the fixed points for n even and $C = \{sr, sr^2, \ldots\}$. Now, for n odd and $C = \{s, sr, sr^2, \ldots\}$, we have, by the q-analogue of Pascal's Identity,

$$\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{q=\xi_2} = \begin{bmatrix} n+k-2 \\ k \end{bmatrix}_{q=\xi_2} + \sum_{i=1}^{k} \begin{bmatrix} n-2+k-i \\ k-i \end{bmatrix}_{q=\xi_2}$$

On the right-hand side, the first term counts the number of fixed k-multisubsets which don't contain the fixed vertex of the reflection and each term in the summation counts the number of fixed k-multisubsets which contain the fixed vertex of the reflection i times. The claim follows.

Now, by an identical computation to the proof of 4, we have

$$\left\{ {n+k-1 \atop k} \right\}_{s=z_1, t=-\det} = Y^{k(n+k-1)} \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{X/Y} = \begin{cases} \left\{ {n+k-1 \atop k} \right\}_{q=\xi_n^\ell} & C = \{r^\ell, r^{n-\ell}\} \\ \left\{ {n+k-1 \atop k} \right\}_{q=\xi_2} & C \text{ reflection conj. class} \end{cases}$$

□

Remark 5.3. A q-analogue of Stanley's hook-content formula

$$s_\lambda(1, q, \ldots, q^{n-1}) = q^{b(\lambda)} \prod_{u \in \lambda} \frac{[n + c(u)]_q}{[h(u)]_q}$$

can be used in the proof of applying 1.7 to proving cyclic sieving for k-multisubsets. In particular, this evaluates to the q-binomial $\begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q$ when letting $\lambda = (k) \vdash k$. For this particular λ, an (s,t) analogue of this evaluates to the Fibonomial for virtually the same reasons giving a "formula"

$$\prod_{u \in \lambda} \frac{\{n + c(u)\}_{s,t}}{\{h(u)\}_{s,t}} = s_\lambda(X^{n-1}, X^{n-2}Y, \ldots, Y^{n-1})$$

where X, Y are defined as above. However, this formula is not always a polynomial in s, t for more general λ, meaning that it is not always possible to take something which has a q-analogue and naturally form the (s,t)-analogue.

6. Dihedral Action on Non-Crossing Partitions of $\{1, \dots, n\}$

$I_2(n)$ has an action on the non-crossing partitions of $[n]$ and also on the non-crossing partitions with a fixed number of blocks, or a fixed rank. The character values for the corresponding representations of both actions were studied by Ding in 2016 in [5]. We show, for odd n, these actions are both instances of dihedral sieving using the natural (s,t)-analogue of the Catalan number as the generating polynomial.

Proposition 6.1. *Let n be odd and $X = \{$non-crossing partitions of $[n]\}$. Then the quadruple*

$$\left(X, \frac{1}{\{n+1\}_{s,t}}\begin{Bmatrix}2n\\ n\end{Bmatrix}_{s,t}, (z_1, -\det), I_2(n)\right)$$

exhibits dihedral sieving.

Proof. Again, let ξ_n be the primitive n-th root of unity $e^{2\pi i/n}$ and let C be a conjugacy class of $I_2(n)$.

First, we compute the character values of $\mathbb{C}^X$. We have that $C_n(\xi_n^\ell)$ counts the fixed points of the action of $\{r^\ell, r^{n-\ell}\}$ by cyclic sieving. Next, we have that $\binom{n}{\lfloor n/2\rfloor}$ is the number of fixed points of $\{r, sr, sr^2, \dots\}$ by Theorem 1 of [4], which is also $C_n(-1)$. This is also Theorem 2.1.5 of [5].

Using the same notation as before, we consider the generalized Catalan number/sequence in Section 5 of [2] $C_{\{n\}_{s,t}} := \frac{1}{\{n+1\}_{s,t}}\left\{\begin{smallmatrix}n\\ k\end{smallmatrix}\right\}_{s,t}$ where we use the specialization $(s,t) = (z_1, -\det)$. Then, we use the fact that

$$\{n+1\}_{s,t} = Y^n[n+1]_{X/Y} = \begin{cases}[n+1]_{\xi_n^\ell} & C \text{ conj. class of } r^\ell\\ [n+1]_{\xi_2} & C \text{ reflection conj. class}\end{cases}$$

to get

$$C_{\{n\}_{(z_1(C), -\det(C))}} = \begin{cases}C_n(q)|_{q=\xi_n^\ell} & C \text{ conj. class of } r^\ell\\ C_n(q)|_{q=\xi_2} & C \text{ reflection conj. class}\end{cases}$$

where $C_{\{n\}}$ is well known to be a polynomial in s, t (in fact, with integral coefficients). $\square$

Proposition 6.2. *Let n be odd and $X = \{$non-crossing partitions of $[n]$ with $n-k$ blocks$\}$. Then the quadruple*

$$\left(X, \frac{1}{\{n\}_{s,t}}\begin{Bmatrix}n\\ k\end{Bmatrix}_{s,t}\begin{Bmatrix}n\\ k+1\end{Bmatrix}_{s,t}, (z_1, -\det), I_2(n)\right)$$

exhibits dihedral sieving.

Proof. First, we show the character values of the $\mathbb{C}^X$ are $N(n,k;\xi_n^\ell)$ for the conjugacy class $\{r^\ell, r^{n-\ell}\}$ and $N(n,k;-1)$ for the conjugacy class $\{r, sr, sr^2, \dots\}$, where $N(n,k) = |X|$ is the *Narayana number* and

$$N(n,k;q) := \frac{1}{[n]_q}\begin{bmatrix}n\\ k\end{bmatrix}_q\begin{bmatrix}n\\ k+1\end{bmatrix}_q q^{k(k+1)} \tag{6.3}$$

the q-analogue of the Narayana number. The character value of rotations is obvious by cyclic sieving, or Theorem 7.2 of [11]. For the case of reflections, the character values are given by $N(n,k;-1)$ by Theorem 3.2.7 of [5].

Now, consider the (s,t)-analogue of the Narayana number. This is a polynomial in s, t for the same reasons $N(n,k;q)$ is a polynomial in q:

$$\begin{aligned}\frac{1}{\{n\}_{s,t}}\begin{Bmatrix}n\\ k\end{Bmatrix}_{s,t} nk+1_{s,t} &= \frac{1}{\{n-k\}_{s,t}}\begin{Bmatrix}n\\ k+1\end{Bmatrix}_{s,t}\begin{Bmatrix}n-1\\ k\end{Bmatrix}_{s,t}\\ &= \frac{1}{\{k+1\}_{s,t}}\begin{Bmatrix}n\\ k\end{Bmatrix}_{s,t}\begin{Bmatrix}n-1\\ k\end{Bmatrix}_{s,t}\\ &= n-1k_{s,t}\begin{Bmatrix}n+1\\ k+1\end{Bmatrix}_{s,t} - nk_{s,t}\begin{Bmatrix}n\\ k+1\end{Bmatrix}_{s,t}\end{aligned}$$

Next, the (s,t)-analogue of the Narayana number becomes, by 3.4,

$$\frac{1}{\{n\}_{s,t}}\begin{Bmatrix}n\\ k\end{Bmatrix}_{s,t}\begin{Bmatrix}n\\ k+1\end{Bmatrix}_{s,t} = Y^{(1-n)+k(n-k)+(k+1)(n-k-1)}N(n,k;X/Y)$$

where the exponent of Y simplifies to $2k(n-k) - 2k$ and if $Y = \xi_n^{-\ell}$, then $n|k\gcd(n,\ell) \implies n|\ell k$ so that the power of Y goes to 1 when the q-binomial $\left[\begin{smallmatrix}n\\ k\end{smallmatrix}\right]_{q=X/Y}$ is non-zero. Thus, the claim follows. $\square$

7. Dihedral Action on Triangulations of a regular n-gon

The case of $I_2(n)$ acting on the triangulations of a regular n-gon is similar to the case of the action of $I_2(n)$ on non-crossing partitions: the character values of $\mathbb{C}^X$ are q-Catalan numbers. The character value of $\mathbb{C}^X$ on the rotation conjugacy class $\{r^\ell, r^{n-\ell}\}$ is $C_{n-2}(q)|_{q=\xi_n^\ell}$ by letting $k = n-3$ in the formula in Theorem 7.1 of [11].

For the case of reflections, an easy counting argument shows, for n odd, the character value of $\mathbb{C}^X$ on $\{s, sr, sr^2, \ldots\}$ is $C_{\frac{n-3}{2}}$ and, for n even, the character on $\{sr, sr^3, \ldots\}$ is 0 and the character on $\{s, sr^2, \ldots\}$ is $2C_{\frac{n-2}{2}}$. Now, we observe

$$nC_{\frac{n-2}{2}} = 2C_{n-2}(-1)$$

for n even. This gives character value $\frac{4}{n}C_{n-2}(-1)$ on $\{s, sr^2, \ldots\}$ for n even and $\frac{2}{n-1}C_{n-3}(-1)$ on $\{s, sr, sr^2, \ldots\}$ for n odd.

We would like to use the (s,t)-analogue of the Catalan number $\frac{1}{\{n+1\}_{s,t}}\left\{{2n \atop n}\right\}_{s,t}$ to prove dihedral sieving for this action for n odd. The key difficulty here is the fact that the reflection character value for the reflection class is a q-Catalan number of order $n-3$ and not $n-2$. Thus, the (s,t)-analogue of the Catalan number fails to work in this case. One would initially expect the case of n even to be easier in this situation since this problem is not apparent. However, other difficulties arise in the n even case as we shall see in 8.

Although the direct (s,t)-analogue of the Catalan number fails to give dihedral sieving, other bivariate analogues of the q-Catalan numbers may prove to be useful here. One of these is Garsia and Haiman's (q,t)-Catalan number $C_n(q,t)$, from [7]. Letting $t \to q^{-1}$ the relation $q^{\binom{n}{2}}C_n(q, q^{-1}) = C_n(q)$ holds. Given that $C_n(q)$ is a polynomial in q, a natural way to split up $C_n(q)$ is by parity of degree. This corresponds to a segregation of terms in $C_n(q,t)$ by total degree of q,t which appears in [8]. One of its special properties is that while $C_n(q,t)$ does not have a unimodal coefficient sequence (with respect to the total degree of t and q), its odd degree and even degree parts do. These parts can be given using the formula

$$\begin{aligned} \text{even degree terms of } C_n(q) &= \frac{1}{2}\left(C_n(q) + C_n(-q)\right) \\ \text{odd degree terms of } C_n(q) &= \frac{1}{2}\left(C_n(q) - C_n(-q)\right) \end{aligned}$$

A possible route to finding dihedral sieving for triangulations (and perhaps for the case of n even for non-crossing partitions) may be to find bivariate analogues for each of these parts rather than finding a bivariate analogue for $C_n(q)$.

8. The case of even n

For the case of n even, $\mathrm{Rep}(I_2(n); \mathbb{C})$ is fundamentally different from the n odd case. In particular, it is likely we require three generators in our presentation of $\mathrm{Rep}(I_2(n); \mathbb{C})$ in order to make our dihedral sieving formulas work. A natural way to extend the (s,t)-polynomials into the trivariate setting is to write

$$\begin{aligned} \{0\}_{s,t,a} &= 0 \\ \{1\}_{s,t,a} &= 1 \\ \{2\}_{s,t,a} &= s \\ \{n\}_{s,t,a} &= \begin{cases} sa\{n-1\} + t\{n-2\} & n \text{ is odd} \\ s\{n-1\} + t\{n-2\} & n \text{ is even.} \end{cases} \end{aligned}$$

Let $b \in \mathrm{Rep}(I_2(n); \mathbb{C})$ be such that b is 0 on every conjugacy class except 4 on $\{s, sr^2, \ldots\}$ and set

$$\begin{aligned} s &= z_1 + \frac{1}{n}b \\ t &= -\det \\ a &= 1 - \frac{1}{4}b. \end{aligned}$$

Then, in fact, the induced trivariate analogue of the Fibonomial $\left\{{n \atop k}\right\}_{s,t,a}$ is a polynomial in s,t,a and evaluates to the same values as in the n odd case. First, we show it is a polynomial in s,t,a.

Theorem 8.1. We have

$$\{m+n\}_{s,t,a} = \begin{cases} a\{m\}\{n+1\} + t\{m-1\}\{n\} & n \text{ odd and } m \text{ even} \\ \{m\}\{n+1\} + at\{m-1\}\{n\} & n \text{ even and } m \text{ odd} \\ \{m\}\{n+1\} + t\{m-1\}\{n\} & m+n \text{ even} \end{cases}$$

Proof. Casework and induction gives

$$\{m+n\} = \begin{cases} sa(\{m\}\{n\}+t\{m-1\}\{n-1\})+t(a\{m\}\{n-1\}+t\{m-1\}\{n-2\}) & n \text{ odd and } m \text{ even} \\ s(a\{m\}\{n\}+t\{m-1\}\{n-1\})+t(\{m\}\{n-1\}+t\{m-1\}\{n-2\}) & n \text{ even and } m \text{ even} \\ sa(\{m\}\{n\}+t\{m-1\}\{n-1\})+t(\{m\}\{n-1\}+at\{m-1\}\{n-2\}) & n \text{ even and } m \text{ odd} \\ s(\{m\}\{n\}+at\{m-1\}\{n-1\})+t(\{m\}\{n-1\}+t\{m-1\}\{n-2\})) & n \text{ odd and } m \text{ odd} \end{cases}$$

An easy inspection shows each of the RHS expressions evaluates to the desired formula. □

The recurrence above immediately gives that the induced Fibonomial-like coefficient is a polynomial in s, t, a.

Theorem 8.2. The induced binomial $\left\{ {n \atop k} \right\}_{s,t,a}$ is a polynomial in s, t, a.

Proof. We use the above theorem to get the result via induction:

$$\left\{ {m+n \atop m} \right\}_{s,t,a} = \begin{cases} a\{n+1\}\left\{ {m+n-1 \atop m-1} \right\} + t\{m-1\}\left\{ {m+n-1 \atop m} \right\} & n \text{ odd and } m \text{ even} \\ \{n+1\}\left\{ {m+n-1 \atop m-1} \right\} + at\{m-1\}\left\{ {m+n-1 \atop m} \right\} & n \text{ even and } m \text{ odd} \\ \{n+1\}\left\{ {m+n-1 \atop m-1} \right\} + t\{m-1\}\left\{ {m+n-1 \atop m} \right\} & m+n \text{ even} \end{cases}$$

□

Now, we study the general solution to the recurrence. Let $b_n = \{2n+1\}$ and let $c_n = \{2n\}$ with $c_0 = 0$ and $b_0 = 1$. We then have

$$b_n = (s^2a+t)b_{n-1} + satc_{n-1}$$

$$c_n = sb_{n-1} + tc_{n-1}$$

Let $B(z) = \sum_{n\geqslant 0} b_n z^n$ and $C(z) = \sum_{n\geqslant 0} c_n z^n$. Then,

$$\frac{B(z)-1}{z} = (s^2a+t)B(z) + satC(z)$$
$$\frac{C(z)}{z} = sB(z) + tC(z)$$

giving

$$B(z) = \frac{1}{1-zs^2a-zt}\left(1+zsatC(z)\right)$$

$$B(z) = \frac{1}{sz}\left(C(z)(1-zt)\right)$$

which gives

$$C(z) = \frac{sz}{z^2t^2+z(-as^2-2t)+1}$$

The quadratic in the denominator has roots

$$R_{1,2} = \frac{as^2+2t\pm\sqrt{(as^2+2t)^2-4t^2}}{2t^2} = \frac{as^2+2t\pm s\sqrt{a^2s^2+4at}}{2t^2} = \begin{cases} \xi_n^{\pm 2\ell} & r^\ell \\ 1 & sr, sr^3, \ldots \\ 1 & s, sr^2, \ldots \end{cases}$$

with the evaluations given when n is even. To solve for c_n, we have

$$C(z) = \frac{sz}{z^2t^2+z(-as^2-2t)+1} = \frac{\frac{-sR_2/t^2}{1-R_2/R_1}}{1-z/R_1} + \frac{\frac{-sR_1/t^2}{1-R_1/R_2}}{1-z/R_2} = \frac{-sR_2/t^2}{1-R_2/R_1}\sum_{n\geqslant 0}\left(\frac{z}{R_1}\right)^n + \left(\frac{-sR_1/t^2}{1-R_1/R_2}\right)\sum_{n\geqslant 0}\left(\frac{z}{R_2}\right)^n$$

so that

$$\{2n\}_{s,t,a} = c_n = \frac{-sR_2/R_1^n}{t^2(1-R_2/R_1)} + \frac{-sR_1/R_2^n}{t^2(1-R_1/R_2)} = \frac{s}{t^2R_1^{n-1}R_2^{n-1}}\left(\frac{R_2^n-R_1^n}{R_2-R_1}\right) = \frac{s}{t^2R_1^{n-1}}[n]_{q=R_1/R_2}$$

Similarly, we have

$$B(z) = \frac{1-zt}{z^2t^2+z(-as^2-2t)+1} = \frac{-\frac{(t-R_2)}{t^2(R_2-R_1)}}{1-z/R_1} + \frac{-\frac{(t-R_1)}{t^2(R_1-R_2)}}{1-z/R_2} = \frac{-(t-R_2)}{t^2(R_2-R_1)}\sum_{n\geqslant 0}\left(\frac{z}{R_1}\right)^n + \frac{(t-R_1)}{t^2(R_1-R_2)}\sum_{n\geqslant 0}\left(\frac{z}{R_2}\right)^n$$

so that

$$\begin{aligned}
\{2n+1\}_{s,t,a} = b_n &= \frac{(t-R_2)/R_1^n}{t^2(R_1-R_2)} - \frac{(t-R_1)/R_2^n}{t^2(R_1-R_2)} \\
&= \frac{R_2^n t - R_2^{n+1} - tR_1^n + R_!^{n+1}}{t^2 R_1^n R_2^n (R_1-R_2)} \\
&= \frac{1}{t^2 R_1^n R_2^n}\left(\frac{R_1^{n+1}-R_2^{n+1}}{R_1-R_2}\right) - \frac{1}{tR_1^n R_2^n}\left(\frac{R_1^n - R_2^n}{R_1-R_2}\right) \\
&= \frac{1}{tR_1^n}\left([n+1]_{q=R_1/R_2} - \frac{1}{tR_2}[n]_{q=R_1/R_2}\right)
\end{aligned}$$

Now, it should be easy to determine the values of $\left\{ n \atop k \right\}_{s,t,a}$ under out substitution of generators of $\mathrm{Rep}(I_2(n);\mathbb{C})$. Although $\left\{ n \atop k \right\}_{s,t,a}$ evaluates to the correct character values for the reflection conjugacy classes under the substitution $(s,t,a) = (z_1 + b/n, -\det, 1 - b/4)$, we have that

$$\left\{ n \atop k \right\}_{s,t,a} = \begin{bmatrix} n \\ k \end{bmatrix}_{q=\xi_n^{2\ell}}$$

in the rotation conjugacy class $\{r^\ell, r^{n-\ell}\}$ which is not always the correct character value. This problem was avoided in the n odd case precisely because $\gcd(n, 2\ell) = \gcd(n, \ell)$.

Remark 8.3. The (s,t,a)-analogue of the Catalan number is a polynomial in s,t,a for the same reasons the (s,t)-analogue of the Catalan number is:

$$\{2n\} = \{n\}\{n+1\} + t\{n-1\}\{n\} \implies \frac{1}{\{n+1\}}\left\{ 2n \atop n \right\} = \left\{ 2n-1 \atop n-1 \right\} + t\left\{ 2n-1 \atop n-2 \right\}$$

Thus, this analogue gives some of the correct values for the noncrossing partitions case with the same exceptions as above.

Another possible generalization is as follows: we define modified (s,t)-analogues

$$\begin{aligned}
\{0\}_{s,t} &= 0 \\
\{1\}_{s,t} &= 1 \\
\{2\}_{s,t} &= 1 + t \\
\{3\}_{s,t} &= 1 + s \\
\{n\}_{s,t} &= s\{n-2\}_{s,t} - \{n-4\}_{s,t}
\end{aligned}$$

Empirically, these seem to satisfy sufficiently nice divisibility properties that $\left\{ n \atop k \right\}_{s,t}$ is always a polynomial. Additionally, the specialization $s = q + q^{-1}$ and $t = q$ gives $\frac{1}{q^{\lfloor n/2 \rfloor}}[n]_q$. Evaluating at the characters of rotations seems to give the numbers of fixed points for k-subsets, up to some sign factor. For n even, evaluating at $s = -2, t = -1$ and $s = -2, t = -1 + \frac{4}{n}$ also seems to give the correct numbers of fixed points up to sign for reflections, and for n odd the same is true for $s = -2, y = \frac{-1}{2}$.

9. Further Questions

A case of dihedral sieving was discovered by Abuzzahab-Korson-Li-Meyer in [1] and later studied in [12]. Let $\lambda = b^a \vdash n$ be a rectangular partition with $ab = n$. Let $\mathrm{SYT}(\lambda)$ be the set of all standard young tableaux of shape λ and let $\mathrm{CST}(\lambda, k)$ be the set of all semi-standard, or column-strict, tableaux of shape λ with entries $\leqslant k$. Both of these sets have an action by $I_2(n)$ as follows: let e act by jeu-de-taquin promotion and let j act by evacuation on these sets. The group $\langle e, j \rangle$ generated by e, j, considered as a subgroup of $S_{\mathrm{CST}(\lambda,k)}$ or $S_{\mathrm{SYT}(\lambda)}$, is isomorphic to a dihedral group of order $2k$. [12] explains that the action of j can be understood in terms of the action of the long cycle of some symmetric group (S_k for the case of $\mathrm{CST}(\lambda, k)$ and S_n for the case of $\mathrm{SYT}(\lambda)$) on some module and that, similarly, the action of e can be understood in terms of the action of a long element (with respect to the Bruhat order). This gives an expression for the fixed point counts of the action in terms of Specht module characters. We first give the character values for the dihedral action on $\mathrm{SYT}(\lambda)$.

Theorem 9.1. [Theorem 1.3 of [12]] Let $\mathbb{Z}/n\mathbb{Z}$ act on $\mathrm{SYT}(\lambda)$ be jeu-de-taquin promotion. Then, the triple $(\mathrm{SYT}(\lambda), \mathbb{Z}/n\mathbb{Z}, X(q))$ exhibits cyclic sieving where $X(q)$ is the q-analogue of the hook length formula

$$X(q) = f^\lambda(q) := \frac{[n]!_q}{\prod_{(i,j)\in\lambda}[h_{ij}]_q}$$

where h_{ij} is the hook-length of cell $(i,j) \in \lambda$.

Theorem 9.2. [Proposition 7.3 of [12]] Let c be the long cycle $(1, 2, \ldots, n)$ in S_n and let $\chi^\lambda : S_n \to \mathbb{C}$ be the irreducible character of S_n corresponding to λ. Let $w_0 \in S_n$ be the long element with respect to the Bruhat order, or the permutation given by $n, n-1, \ldots, 1$ in one-line notation. Let $\langle e, j\rangle \subset S_{\mathrm{SYT}(\lambda)}$ be as above. Let $\phi : \langle w_0, c\rangle \to \langle e, j\rangle$ be defined by $\phi(w_0) = e$ and $\phi(c) = j$ so that it is a group epimorphism. For any $g \in \langle e, j\rangle$ and any w such that $\phi(w) = g$, we have

$$|\,\mathrm{SYT}(\lambda)^g| = \pm\chi^\lambda(w) \tag{9.3}$$

It would be interesting if the above theorems could be rephased in the representation-theoretic formulation of dihedral sieving given in 3. Unfortunately, even for rectangular partitions, the known formulas for the Specht module character χ_λ are difficult to work with. Next, for $\mathrm{CST}(\lambda, k)$, we have the following.

Theorem 9.4. Let $\mathbb{Z}/k\mathbb{Z}$ act on $\mathrm{CST}(\lambda, k)$ via jeu-de-taquin promotion. Then, the triple $(\mathrm{CST}(\lambda, k), \mathbb{Z}/k\mathbb{Z}, X(q))$ exhibits cyclic sieving where $X(q)$ is the q-shift of the principal specialization of the Schur function

$$X(q) := q^{-\kappa(\lambda)} s_\lambda(1, q, q^2, \ldots, q^{k-1})$$

where κ is the statistic on partitions $\lambda = (\lambda_1, \lambda_2, \ldots)$ given by

$$\kappa(\lambda) = 0\lambda_1 + 1\lambda_2 + 2\lambda_3 + \cdots$$

Theorem 9.5. [Proposition 7.6 of [12]] If k is odd, we have that

$$|\,\mathrm{CST}(\lambda, k)^e| = (-1)^{\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-1}) = |\,\mathrm{CST}(\lambda, k)^{ej}| \tag{9.6}$$

If k is even, we have that

$$|\,\mathrm{CST}(\lambda, k)^e| = (-1)^{\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-1}) \tag{9.7}$$

$$|\,\mathrm{CST}(\lambda, k)^{ej}| = \begin{cases} (-1)^{\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-3}, (-1)^{k-2}, (-1)^{k-2}) & a \text{ and } b \text{ even} \\ (-1)^{\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-1}) & a \text{ even and } b \text{ odd} \\ (-1)^{b/2-1+\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-1}) & a \text{ odd and } b \text{ even} \\ (-1)^{\lfloor b/2\rfloor+\kappa(\lambda)} s_\lambda(1, -1, \ldots, (-1)^{k-3}, (-1)^{k-2}, (-1)^{k-2}) & a \text{ and } b \text{ odd} \end{cases}$$

For the case of $\mathrm{CST}(\lambda, k)$ and k odd, the character values take the same form as the character values in our proven instances of dihedral sieving (namely, the cyclic sieving polynomial evaluated at $q = -1$ gives the fixed point count for the reflection conjugacy class). However, a dihedral sieving polynomial cannot be obtained from 5.1 since the specialization is shifted by a power of q. In fact, the proof of Theorem 1.4 in [12] does not even follow immediately from 1.7.

Another possible instance of dihedral sieving arises in the study of alternating sign matrices. An alternating sign matrix (ASM) of size n is an $n \times n$ matrix with all entries $0, 1, -1$ such that the non-zero entries in each row and column alternate sign and sum to 1. Let $\mathrm{ASM}(n)$ be the set of all alternating sign matrices of size n. Then, Zeilberger showed in [13] that

$$|\,\mathrm{ASM}(n)| = \prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$$

The natural q-analogue of this formula

$$X(q) := \prod_{k=0}^{n-1} \frac{[3k+1]!_q}{[n+k]!_q}$$

is a polynomial in q. The cyclic group $\mathbb{Z}/4\mathbb{Z}$ acts on $\mathrm{ASM}(n)$ by rotations through 90° so that $(\mathrm{ASM}(n), \mathbb{Z}/4\mathbb{Z}, X(q))$ exhibits cyclic sieving. However, there is no linear algebraic proof of this statement and, rather, $X(q)$ is only known to be a generating function for descending plane partitions a related combinatorial object, by weight. The dihedral group $I_2(4)$ also acts on $\mathrm{ASM}(n)$. The fixed-point count for a reflection through a vertical axis (or for that of its conjugate, a reflection through a horizontal axis) was found by Kuperberg in [10] to be

$$(-3)^{n^2} \prod_{i,j \leqslant 2n+1, 2|j} \frac{3(j-i)+1}{j-i+2n+1}$$

when the size of the matrix is $(2n+1) \times (2n+1)$ (there are no $(2n) \times (2n)$ alternating sign matrices fixed by a reflection of this type). However, the case of reflections through a diagonal axis is harder: no formula for the fixed-point count is currently known or conjectured. However, given recent interest in this subject, a dihedral-sieving interpretation of this problem might give insight into the cyclic sieving case and into the enumeration of alternating sign matrices invariant under certain dihedral symmetries.

10. Appendix: Character Tables

n even	e	$r^\ell, r^{n-\ell}$	$sr, sr^3, sr^5, \ldots$	$s, sr^2, sr^4, \ldots$
$\mathbb{1}$	1	1	1	1
$\det$	1	1	-1	-1
$\chi_b := \begin{cases} \langle r^2, s\rangle \mapsto 1 \\ r \mapsto -1 \end{cases}$	1	$(-1)^\ell$	-1	1
$\chi_b \cdot \det$	1	$(-1)^\ell$	1	-1
χ_m	2	$2\cos\left(\frac{2\pi m\ell}{n}\right)$	0	0
$\chi_{k\text{-subsets}}$	$\binom{n}{k}$	$\genfrac{[}{]}{0pt}{}{n}{k}_{q=\xi_n^\ell}$	$\genfrac{[}{]}{0pt}{}{n}{k}_{q=\xi_2}$	$\genfrac{[}{]}{0pt}{}{n}{k}_{q=\xi_2} + 2\genfrac{[}{]}{0pt}{}{n-2}{k-1}_{q=\xi_2} + \genfrac{[}{]}{0pt}{}{n-2}{k-2}_{q=\xi_2}$
$\chi_{\mathrm{NC}(n)}$	C_n	$C_n(q)\vert_{q=\xi_n^\ell}$	$C_n(q)\vert_{q=\xi_2}$	$C_n(q)\vert_{q=\xi_2}$
$\chi_{\text{triangulations}}$	C_{n-2}	$C_{n-2}(q)\vert_{q=\xi_n^\ell}$	0	$\frac{4}{n}C_{n-2}(q)\vert_{q=\xi_2}$
$\chi_{\mathrm{NC}(n,k)}$	$N(n,k)$	$N(n,k,\xi_n^\ell)$	$N(n,k,\xi_2)$	$N(n,k,\xi_2)$
$\chi_{k\text{-multisubsets}}$	$\binom{n+k-1}{k}$	$\genfrac{[}{]}{0pt}{}{n+k-1}{k}_{q=\xi_n^\ell}$	$\genfrac{[}{]}{0pt}{}{n+k-1}{k}_{q=\xi_2}$	$\genfrac{[}{]}{0pt}{}{n+k-2}{k}_{q=\xi_2} + 2\genfrac{[}{]}{0pt}{}{n+k-3}{k-1}_{q=\xi_2} + \genfrac{[}{]}{0pt}{}{n-3+k}{k-2}_{q=\xi_2}$

n odd	e	$r^\ell, r^{n-\ell}$	$sr, sr^2, sr^3, \ldots$
$\mathbb{1}$	1	1	1
$\det$	1	1	-1
χ_m	2	$2\cos\left(\frac{2\pi m\ell}{n}\right)$	0
$\chi_{k\text{-subsets}}$	$\binom{n}{k}$	$\genfrac{[}{]}{0pt}{}{n}{k}_{q=\xi_n^\ell}$	$\genfrac{[}{]}{0pt}{}{n}{k}_{q=\xi_2}$
$\chi_{\mathrm{NC}(n)}$	C_n	$C_n(q)\vert_{q=\xi_n^\ell}$	$C_n(q)\vert_{q=\xi_2}$
$\chi_{\text{triangulations}}$	C_{n-2}	$C_{n-2}(q)\vert_{q=\xi_n^\ell}$	$\frac{2}{n-1}C_{n-3}(q)\vert_{q=\xi_2}$
$\chi_{NC(n,k)}$	$N(n,k)$	$N(n,k;\xi_n^\ell)$	$N(n,k;\xi_2)$
$\chi_{k\text{-multisubsets}}$	$\binom{n+k-1}{k}$	$\genfrac{[}{]}{0pt}{}{n+k-1}{k}_{q=\xi_n^\ell}$	$\genfrac{[}{]}{0pt}{}{n+k-1}{k}_{q=\xi_2}$

Acknowledgments

This research was carried out as part of the 2017 summer REU program at the School of Mathematics, University of Minnesota, Twin Cities, and was supported by NSF RTG grant DMS-1148634. The authors would like to thank Victor Reiner, Pavlo Pylyavskyy, and Benjamin Strasser for their mentorship and support.

References

[1] Omar Abuzzahab, Matthew Korson, Martin Li, and Seth Meyer. "Cyclic and dihedral sieving for plane partitions". In: *U. Minnesota REU Report* (2005). URL: `http://www-users.math.umn.edu/~reiner/REU/PlanePartitionReport.pdf`.

[2] Tewodros Amdeberhan, Xi Chen, Victor H. Moll, and Bruce E. Sagan. "Generalized Fibonacci polynomials and Fibonomial coefficients". In: *Annals of Combinatorics* 18.4 (2013), pp. 541–562. URL: `https://arxiv.org/pdf/1306.6511.pdf`.

[3] Helene Barcelo, Victor Reiner, and Dennis Stanton. "BiMahonian Distributions". In: *J. London Math. Soc* 2.77 (2008), pp. 627–646. URL: `https://arxiv.org/pdf/math/0703479.pdf%7D`.

[4] David Callan and Len Smiley. "Noncrossing Partitions Under Rotation and Reflection". In: 2005.

[5] Ziqian Ding. "Dihedral Symmetries of Non-crossing Partition Lattices". PhD thesis. 2016. URL: `http://scholarlyrepository.miami.edu/cgi/viewcontent.cgi?article=2698&context=oa_dissertations`.

[6] Christian Gaetz. "Critical Groups of McKay-Cartan Matrices". In: 2016. URL: `http://www.mit.edu/~gaetz/docs/final_thesis.pdf`.

[7] Adriano M. Garsia and Mark Haiman. "A remarkable (q,t)-Catalan sequence and q-Lagrange inversion". In: *Journal of Algebraic Combinatorics* 5.3 (July 1996), pp. 191–244. DOI: `10.1007/BF00243787`. URL: `https://doi.org/10.1007/BF00243787`.

[8] Mark Haiman. "Conjectures on the Quotient Ring by Diagonal Invariants". In: *Journal of Algebraic Combinatorics* 3.1 (Jan. 1994), pp. 17–76. ISSN: 1572-9192. DOI: `10.1023/A:1022450120589`. URL: `https://doi.org/10.1023/A:1022450120589`.

[9] Verner E. Hogatt Jr. and Calvin T. Long. "Divisibility properties of generalized Fibonacci polynomials". In: *Fibonacci Quarterly* 12 (1974), pp. 113–120.

[10] Greg Kuperberg. "Symmetry Classes of Alternating-Sign Matrices under One Roof". In: *Annals of Mathematics* 156.3 (2002), pp. 835–866. ISSN: 0003486X. URL: `http://www.jstor.org/stable/3597283`.

[11] Victor Reiner, Dennis Stanton, and Dennis White. "The Cyclic Sieving Phenomenon". In: *Journal of Combinatorial Theory* 108.1 (2004), pp. 17–50. URL: `http://www-users.math.umn.edu/~reiner/REU/REU2017notes/ReinerStantonWhite.pdf`.

[12] Brendon Rhoades. “Cyclic sieving, promotion, and representation theory”. In: *Journal of Combinatorial Theory Series A* 117.1 (2010), pp. 38–76. ISSN: 0097-3165. DOI: http://dx.doi.org/10.1016/j.jcta.2009.03.017. URL: http://www.sciencedirect.com/science/article/pii/S0097316509000703.

[13] Doron Zeilberger. “Proof of the Alternating Sign Matrix Conjecture”. In: *ArXiv Mathematics e-prints* (July 1994). eprint: math/9407211.