Generalizing Rules of Three

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1 Introduction

This paper explores *The Rule of Three*, the phenomenon where for some sequences in a ring, all elements in the sequence satisfy a commutation relation precisely when the relation is satisfied by subsets of size three and fewer.

This behavior has been seen in some notable cases. Kirillov [3] shows that elementary symmetric polynomials in noncommuting variables commute (and, in some cases, all Schur functions) when elementary symmetric polynomials of degree at most three commute when restricted to at most three of the variables. Generalizing this, Blasiak and Fomin [1] give a wider theory for rules of three of generating functions over rings via rules of three in sums and products.

In this report we detail our attempts to apply the general theory to interesting specific cases. Namely, we consider Schur Q-functions and loop symmetric functions. In the former case, we give a conjecture for a rule of three, and give progress towards a proof for the said conjecture, emulating the structure of the proof given for super elementary symmetric functions in [1]. In the latter case, we find negative results.

2 Background

First, we give some notation. Let e_k and h_k denote the kth elementary symmetric polynomial and kth homogeneous symmetric polynomial, respectively. We use the standard definition, in

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both the commutative and noncommutative sense:

$$e_k(x_1, \cdots, x_n) = \sum_{1 \le j_1 < \cdots < j_k \le n} x_{j_k} \cdots x_{j_1}$$
$$h_k(x_1, \cdots, x_n) = \sum_{1 \le j_1 \le \cdots \le j_k \le n} x_{j_1} \cdots x_{j_k}$$

For $x_1, \ldots, x_n$ and $S \subset [n]$ with $S = \{s_1 < s_2 < \cdots < s_k\}$, let $e_i(x_S)$ denote the polynomial with x_{s_1} through x_{s_k} as variables. However, when speaking of x_S in other contexts, it will denote the product $x_{s_k} x_{s_{k-1}} \cdots x_{s_1}$. Note that we multiply in descending order; when we want to explicitly give the order for multiplying we will add an arrow in the superscript. For example, $x_S^{\uparrow}$ will be multiplying in ascending order. Finally, let $[x, y] = xy - yx$ be the commutator in the standard sense.

[6] gives some definitions that we reproduce here. A *strict partition* of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in \mathbb{Z}^\ell$ such that $\lambda_1 > \lambda_2 > \cdots > \lambda_\ell$. The corresponding *shifted diagram* of λ is an array of square cells in which the ith row has λ_i cells, and is shifted $i - 1$ units to the right with respect to the top row.

A *(semistandard) shifted Young tableau* T of shape λ is a filling of a shifted diagram λ with letters from the alphabet $A = \{1' < 1 < 2' < 2 < \cdots\}$ such that:

- Rows and columns are weakly increasing;
- Each column has at most one k for $k \in \{1, 2, \cdots\}$;
- Each row has at most one k for $k \in \{1', 2', \cdots\}$.

These allow us to define the Schur Q-functions Q_λ for a shifted partition λ.

Definition. $Q_\lambda(x_1, \ldots, x_n) \in \mathbb{Q}[x_1, \ldots, x_n]$ can be defined in any of the following equivalent ways: [7]

(a) For a tableau T with content $(a_1, a_2, \ldots)$ (we ignore the distinction between primed and unprimed entries here), define $x^T = x_1^{a_1} x_2^{a_2} \cdots$. Then $Q_\lambda = \sum_{\text{shape}(T)=\lambda} x^T$, where we sum over shifted tableaus T.

(b) Let q_k be the kth coefficient of the product of generating functions $\prod_{i=1}^n f_i$, where

$$f_i = \frac{1 + x_i t}{1 - x_i t} = 1 + 2(x_i t + x_i^2 t^2 + \cdots)$$

And let

$$Q_{(k_1,k_2)} = q_{k_1} q_{k_2} + \sum_{i=1}^{k_2} (-1)^i q_{k_1+i} q_{k_2-i}$$

Then for λ with ℓ parts,

$$Q_\lambda = \begin{cases} \mathrm{pf}[Q_{(\lambda_i,\lambda_j)}]_{1 \le i < j \le \ell} & \ell \text{ even} \\ \mathrm{pf}[Q_{(\lambda_i,\lambda_j)}]_{1 \le i < j \le \ell+1} & \ell \text{ odd} \end{cases}$$

where pf is the Pfaffian, and $\lambda_{\ell+1} = 0$ when ℓ is odd.

These are symmetric (although that is not obvious), and in fact, the Schur Q-functions generate a subalgebra of the space of symmetric functions, namely ([5] 2.5)

$$\mathbb{Q}[q_k] = \mathbb{Q}[p_{2k+1}] \qquad \text{where } p_k = \sum_i x_i^k$$

Further, ([5] 2.6)

$$q_k = \sum_{\substack{(i_1,i_3,\ldots,i_p) \\ k=i_1+3i_3+\ldots+pi_p}} \frac{2^{i_1+\cdots+i_p}}{i_1!\cdots i_p!}\left(\frac{p_1}{1}\right)^{i_1}\left(\frac{p_p}{p}\right)^{i_p}$$

So this immediately gives us that q_{2k+1} is an algebraically independent generating set of Λ (at least in the commutative case).

3 Results

3.1 Schur-Q functions, noncommutative case hook reading

To move Schur Q-functions to the noncommutative case, we need to decide on a method for reading the tableau. Following [6], we will use the hook reading, which preserves elementary properties of commutative Schur Q-functions. In this context, $Q_{(k)}$, which we will refer to as q_k, is defined as follows:

Definition. $q_k(x_1, \ldots, x_n) \in \mathbb{Q}\langle x_1, \ldots, x_n\rangle$ can be defined in any of the following equivalent ways:

(a) The standard tableaux definition of q_k, except x^T is defined by reading the unprimed entries right to left, then the primed entries left to right. For example, if

$$T = \boxed{1}\boxed{1'}\boxed{2'}\boxed{4}\boxed{5'}\boxed{5'}\boxed{6}$$

then $x^T = (x_6x_4x_1)(x_1x_2x_5x_5)$.

(b) Let $g_i = 1 + x_it$ and let $h_i = (1 - x_it)^{-1}$. q_k is the kth coefficient of the product of generating functions

$$\prod_{i=n}^{1} g_i \prod_{i=n}^{1} h_i$$

(c) q_k is the sum of monomials whose subscripts strictly decrease, then nonstrictly increase, with a factor of two added.

Evidence from Sage suggests that a rule of 3 is possible. Namely, with three commuting variables, $[q_3, q_5]$ is contained in the ideal generated by all of the relations only using q_k for k at most three. Further, with two commuting variables, all relations of degree at most $k+1$ (that is, $[q_i, q_]$ for $i + j \leq k + 1$) are contained in the ideal generated by the relations involving q_i and q_1 for i at most k. Notice that we only need to worry about q_k for k odd, since commutation relations hold for even k when they hold for odd k by 2.

Consider some noncommuting variables $x_1, \ldots, x_N$, and notate the elements of $S \subset [N]$ by $S = \{s_1 < s_2 < \cdots < s_n\}$. Then we define $q_k(x_S)$ as the Schur Q-function in n variables where x_i is replaced with x_{s_i}. We conjecture the following:

Conjecture 3.1. *Let $x_1, \ldots, x_N, y_1, \ldots, y_N$ be elements of a ring A. The following are equivalent:*

- *$q_k(x_S)$ and $q_\ell(y_S)$ commute for all S, k, ℓ.*
- *the above holds when $k = 1$ or $\ell = 1$, and for all S*

We have found no evidence to support anything stronger; the statement does not hold when the second case is restricted to $|S| \leq 3$, and it does not hold when the second case specifies $kl \leq 5$.

We can rewrite this in terms of generating functions, as done in [1], most similarly to Lemma 8.2 in the cited paper.

Conjecture 3.2. *Let A be a ring, and let $x_1, \ldots, x_N, y_1, \ldots, y_N \in A$ satisfy*

$$[x_a, y_b] = [y_a, x_b] \qquad \text{for all } 1 \leq a \leq b \leq N$$

Further, let

$$\begin{aligned} a_i &= 1 + x_i t & \qquad b_i &= 1 - x_i t \\ \alpha_i &= 1 + y_i s & \qquad \beta_i &= 1 - y_i s \end{aligned}$$

Notice that this implies that all expresssions of the same index commute: $x_i, y_i, a_i, b_i, \alpha_i, \beta_i$ commute with each other (along with their inverses). Further, suppose that the following relations are satisfied:

$$\left[\sum_{i \in S} x_i, \alpha_S(\beta_S)^{-1}\right] = 0$$
$$\left[\sum_{i \in S} y_i, a_S(b_S)^{-1}\right] = 0$$

Then $a_{[N]}(b_{[N]})^{-1}\alpha_{[N]}(\beta_{[N]})^{-1} = \alpha_{[N]}(\beta_{[N]})^{-1}a_{[N]}(b_{[N]})^{-1}$.

We would like to give this statement without explicitly defining a_i, b_i, α_i, and β_i, instead giving only the relations that are necessary for the proof. However, we do not know the full list of relations necessary. This is covered more in 3.5.

We can prove that rephrasing the conjecture in terms of generating functions indeed gives us an equivalent statement.

Theorem 3.3. *3.1 and 3.2 are equivalent.*

Proof. Using $a_i, b_i, \alpha_i, \beta_i$ as specified, the two relations in 3.1 are equivalent as follows:

$$\left[\sum_{i\in S} x_i, \alpha_S(\beta_S)^{-1}\right] = \left[q_1(x_S), \sum_{i\geq 0} q_i(y_S)s^i\right] = 0 \iff [q_1(x_S), q_i(y_S)] = 0 \quad \forall\, i$$
$$\left[\sum_{i\in S} y_i, a_S(b_S)^{-1}\right] = \left[q_1(y_S), \sum_{i\geq 0} q_i(x_S)t^i\right] = 0 \iff [q_1(y_S), q_i(x_S)] = 0 \quad \forall\, i$$

The $[x_a, y_b] = [y_a, x_b]$ condition is implied by $[q_1, q_1] = 0$ for $|S| \leq 2$. Further,

$$\left[a_{[n]}(b_{[n]})^{-1}, \alpha_{[n]}(\beta_{[n]})^{-1}\right] = \left[\sum_{i\geq 0} q_i(x_S)t^i, \sum_{j\geq 0} q_j(y_S)s^j\right] = 0$$
$$\text{if and only if } [q_i(x_{[n]}), q_j(y_{[n]})] = 0 \quad \forall\, i, j$$

While this only gives us the statement for the set of variables indexed by $[n]$, the set of conditions hold for all $S \subset [n]$, so we can restrict to some T and get the statement for every subset as well. $\square$

We can reproduce some lemmas that act as an analogue for those in the proof of Lemma 8.2 in [1].

Lemma 3.4 (6.1 analogue)**.** Let R be a ring, and let $x_i, x_j, \alpha_i, \beta_i, \alpha_j, \beta_j$ satisfy the following conditions:

- all elements of the same index commute
- $[x_i + x_j, \alpha_j\alpha_i\beta_i\beta_j] = 0$

Then $(\alpha_j[\alpha_i, x_j] - [x_i, \alpha_j]\alpha_i)\beta_i\beta_j = \alpha_j\alpha_i([x_j, \beta_i]\beta_j - \beta_i[\beta_j, x_i])$.

Remark 3.5. In the proof of Blasiak-Fomin 8.2, this lemma is enough to prove the statement for $N = 2$. However, considering the two equations we get from applying 3.4 to the relations from 3.2:

$$(\alpha_2[\alpha_1, x_2] - [x_1, \alpha_2]\alpha_1)\beta_1^{-1}\beta_2^{-1} = \alpha_2\alpha_1([x_2, \beta_1^{-1}]\beta_2^{-1} - \beta_1^{-1}[\beta_2^{-1}, x_1])$$
$$(a_2[a_1, y_2] - [y_1, a_2]a_1)b_1^{-1}b_2^{-1} = a_2a_1([y_2, b_1^{-1}]b_2^{-1} - b_1^{-1}[b_2^{-1}, y_1])$$

Sage computation confirms that these relations, along with the commutation relations given in 3.2, are not enough to prove that $a_2a_1b_1b_2$ and $\alpha_2\alpha_1\beta_1\beta_2$ commute. Thus, an attempt at a proof must use more than what is presented in this paper; we suspect this may involve a relation in bs and βs, since we have a relation among as and αs; arithmetic suggests that having more analogous relations would give progress towards the desired result.

This also shows that we cannot generalize 3.2 to arbitrary a_i, b_i, α_i, and β_i, as is done in Lemma 8.2 in Blasiak-Fomin. We will need to find the complete list of relations necessary in the specific case to do so.

We have also considered the weakened version of allowing x_i and x_j to commute when $|i-j|>1$, as is done in [2].

Conjecture 3.6. *3.1 and 3.2 hold when* $[x_i,x_j]=[x_i,y_j]=[y_i,y_j]=0$ *for* $|i-j|>1$.

This allows us to give an analogue to 6.4.

Lemma 3.7 (6.4 analogue)**.** Suppose:

$$\left[\sum_{i\in S} x_i, \alpha_S^{\downarrow}\beta_S^{\uparrow}\right]=0 \text{ for all } S\subset\{1,\dots,N\}$$

And suppose $[x_i,\alpha_k]=[x_i,\beta_k]=0$ for nonadjacent i,k. Then, for any $S=\{s_1<\cdots<s_m\}$,

$$([x_{s_{m-1}},\alpha_{s_m}]\alpha_{s_{m-1}}-\alpha_{s_m}[\alpha_{s_{m-1}},x_{s_m}])\alpha_U\beta_S=\alpha_S\beta_U(\beta_{s_{m-1}}[\beta_{s_m},x_{s_{m-1}}]-[x_{s_m},\beta_{s_{m-1}}]\beta_{s_m})$$

Proof. Let $T=S\setminus s_m$ and let $U=S\setminus\{s_m,s_{m-1}\}$. Then

$$\begin{aligned}
\left[\sum_{i\in S} x_i, \alpha_S^{\downarrow}\beta_S^{\uparrow}\right] &= [x_{s_m},\alpha_{s_m}\alpha_T\beta_T\beta_{s_m}]+\left[\sum_{i\in T} x_i,\alpha_{s_m}\alpha_T\beta_T\beta_{s_m}\right]\\
&=\alpha_{s_m}[x_{s_m},\alpha_T\beta_T]\beta_{s_m}+\left[\sum_{i\in T} x_i,\alpha_{s_m}\right]\alpha_T\beta_S+\alpha_S\beta_T\left[\sum_{i\in T} x_i,\beta_{s_m}\right]\\
&=\alpha_{s_m}[x_{s_m},\alpha_T\beta_T]\beta_{s_m}+[x_{s_{m-1}},\alpha_{s_m}]\alpha_T\beta_S+\alpha_S\beta_T\left[x_{s_{m-1}},\beta_{s_m}\right]
\end{aligned}$$

And the statement follows. ∎

3.2 Negative Results

When generalizing Schur Q-functions to the noncommutative case, we can also consider the less natural choice of reading shifted tableaus in descending order from right to left. This is equivalent to using the generating polynomial f in the non-commutative setting.

That is, define q_k in the non-commutative case as the kth coefficient of the product of generating functions $\prod f_i$ from 1 to n, where

$$f_i=(1+x_it)(1-x_it)^{-1}$$

In this case, we immediately get the following result from Theorem 2.5 of [1]:

Corollary 3.8. *The following are equivalent:*

- $[q_k(u_S),q_\ell(u_S)]=0$ *for any* $k,\ell,S\subset N$
- *the above holds for* $|S|=2,3$.

However, this theorem, unlike typical rules of three, does not allow for two sets of non-commuting variables. Notice that Conjecture 2.3 of [1], if proved, would allow for two sets of non-commuting variables.

Also notice that this corollary has no restriction on k and ℓ. Computational evidence suggests that we cannot restrict $k, \ell \leq 5$, and we also cannot restrict to either k or ℓ being 1.

We define loop elementary symmetric functions on n flavors through the generating function called the whirl matrix, [4]

$$\begin{bmatrix} 1 & a_i^{(1)} & 0 & \cdots & 0 \\ 0 & 1 & a_i^{(2)} & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_i^{(n)} t & 0 & 0 & \cdots & 1 \end{bmatrix}$$

The product of these are

$$P_{ij} = \sum_{k=0}^{\left\lfloor \frac{i-j+m}{n} \right\rfloor} e^{(i)}_{kn-(i-j)} t^k$$

where the superscript on e is the flavor of the first element in all of the monomials, n is the number of flavors, and m is the number of generating functions in the product.

Computing the three-flavor three-variable case suggests that a rule of three for these functions does not exist: when degree one loop symmetric polynomials commute with degree two and degree three polynomials, we do not get commutation between degree two and degree three polynomials. Because we are working with only one set of non-commuting variables, this has been true for all of the rules of three mentioned in this paper.

4 Further Directions

In following the form of Blasiak and Fomin's proof of Lemma 8.2, we have achieved the following in the standard case and in the weakened case (where nonadjacent variables commute).

$$\begin{tikzcd}
 & 6.1 \arrow[d] \arrow[dr] & 6.2 \arrow[d] & 6.3 \arrow[dl] \\
8.1 \arrow[r] \arrow[dr] & \triangle & 6.4 \arrow[dl] & \\
 & \square & &
\end{tikzcd}$$

In the diagram, $\triangle$ represents the $N = 2$ case, and $\square$ represents a full proof. Thus, any progress on 8.1 would result in a significant result.

Beyond proving the conjecture, we can also ask when commutativity of q_k, the elementary Schur Q-functions, extends to all Schur Q-functions, as is explored in [2]. In the weakened case, this phenomenon may occur, since the question reduces to deciding whether the Pfaffian formula holds true in the noncommutative case.

A Code

All code mentioned is located at: `https://github.com/ewin-t/ruleof3qschur`

The README in the repository explains the precise content of the code.

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