Cluster Variables and Perfect Matchings of Subgraphs of the dP_3 Lattice

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Abstract

We give a combinatorial intepretation of cluster variables of a specific cluster algebra under a mutation sequence of period 6, in terms of perfect matchings of subgraphs of the brane tiling dual to the quiver associated with the cluster algebra.

1 Introduction

Cluster algebras, introduced by Fomin and Zelevinsky in the past decade [4, 5], arise natually in diverse areas of mathematics such as total positivity, tropical geometry and Lie theory. Previous work, such as [1, 9, 10, 11], gave combinatorial intepretations for the cluster variables as perfect matchings of graphs, under suitable weighting schemes. Of particular interest is the situation where the graphs are directly related to the quiver of the cluster algebra, namely when they are subgraphs of the dual of the quiver. Some motivating examples include Speyer's proof of the Aztec Diamond Theorem [13] and Jeong's Tree Phenomenon and Superposition Phenomenon [7].

In this report we give a specific example of cluster variables whose Laurent expansions can be intepreted as perfect matchings of subgraphs of the brane tiling dual to the original quiver.

2 Quiver, dP_3 Lattice and Cluster Variables

Figure 1 shows the quiver Q we are working with, which is listed in [6] as Model 10 Phase a. The *brane tiling* (doubly periodic planar bipartite graph)

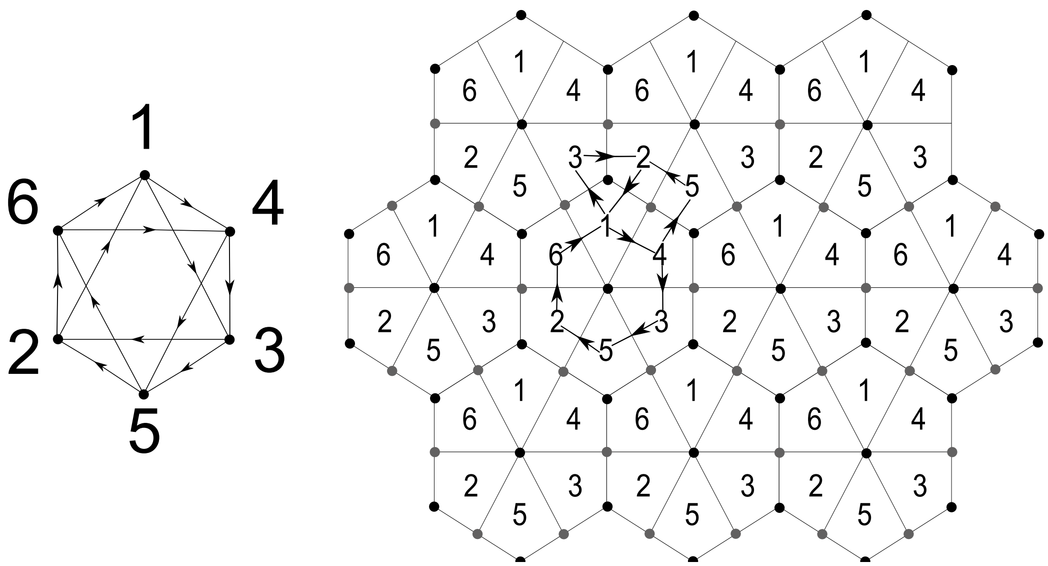

Figure 1: Quiver Q and its brane tiling, dP_3 lattice

dual to Q is a superposition of the triangular lattice with its dual hexagonal lattice; we follow [3] and call it the dP_3 lattice.

Definition. Let $\sigma := (15)(24)(36)$, a permutation of 6 numbers. It corresponds to $180°$ rotation of Q or the lattice.

Mutation at node a and at node $\sigma(a)$ commutes since there are no arrows between the two nodes. The two mutations preformed as a pair reverse arrows incident to a and $\sigma(a)$. Therefore after cyclically mutating all 3 pairs of nodes, we will get back Q, i.e. Q has period 6. Consider the periodic mutation sequence at vertices $2, 4, 5, 1, 3, 6, \ldots$ (Figure 2). Name the new cluster variables by:

$$\xrightarrow{\mu_2} y_1 \xrightarrow{\mu_4} y_1' \xrightarrow{\mu_5} y_2 \xrightarrow{\mu_1} y_2' \xrightarrow{\mu_3} y_3 \xrightarrow{\mu_6} y_3' \xrightarrow{\mu_2} \ldots$$

By symmetry of Q under σ, $y_N' = \sigma(y_N)$, where σ acts naturally on Laurent polynomials via $\sigma(x_i) = x_{\sigma(i)}$.

For consistency, set $y_{-2} = x_2$, $y_{-2}' = x_4$, $y_{-1} = x_5$, $y_{-1}' = x_1$, $y_0 = x_3$, $y_0' = x_6$. The exchange relation now becomes:

$$y_N y_{N-3} = y_{N-1} y_{N-2} + y_{N-1}' y_{N-2}' \tag{1}$$

for $N \geq 1$.

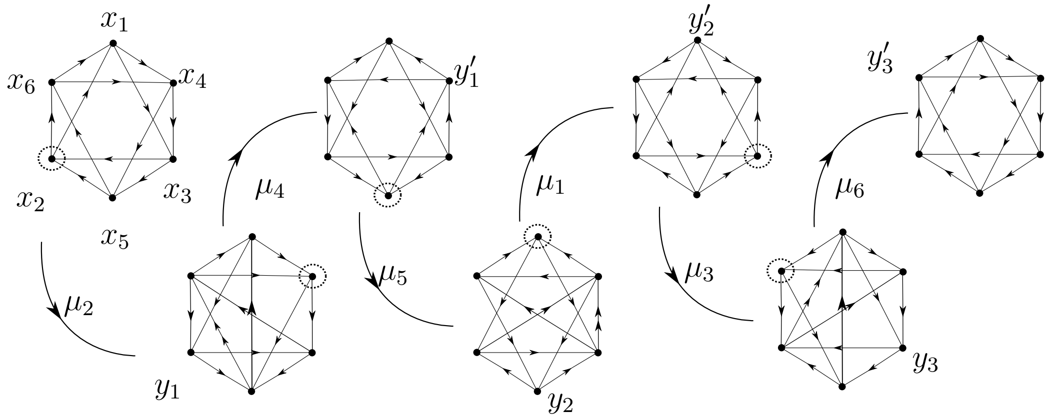

Figure 2: Quiver Q under the periodic mutation sequence $2, 4, 5, 1, 3, 6, \ldots$

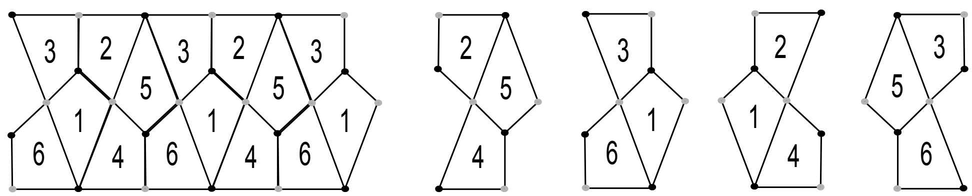

Figure 3: A strip and four types of blocks

3 Diamonds on the dP_3 lattice

We will discuss a sequence of subgraphs on the dP_3 lattice, first introduced by Propp [12], further studied by Ciucu under the name of "Aztec Dragons" [2], and recently extended to half-integral order by Cottrell and Young [3], in which they are called diamonds.

Definition. A *strip* is a subgraph of dP_3 bounded between two neighboring horizontal lines. A *block* is one of the four following graphs bounding the union of 3 adjacent faces: [254], [316], [214], [356] (Figure 3).

Fix a strip s_0, let s_j be the j-th strip north (resp. south) of s_0 if $j > 0$ (resp. $j < 0$). For $j \leq 0$, tile s_j alternatingly by blocks [254] and [316], while for $j > 0$, tile s_j by blocks [214] and [356]. Note that each block is adjacent to one block in each of the 4 directions. Label each block $T(i, j)$ such that $T(0, 0)$ is [254], block $T(i, j)$ is in strip s_j, directly east of block $T(i - 1, j)$ and north of $T(i, j - 1)$.

Definition. For $n \in \mathbb{Z}_{>0}$, a *diamond* of order n is

$$D_n = \bigcup_{|i+n-1|+|j| \leq n-1} T(i,j)$$

and a *diamond* of order $n+1/2$ is

$$D_{n+1/2} = D_n \bigcup \left(\bigcup_{\substack{-n+2 \leq i \leq 1 \\ 1 \leq j \leq 2-i}} T(i,j) \right) \bigcup S_3 \bigcup S_2$$

where S_3 (resp. S_2) is the square labeled 3 (resp. 2) in the block $T(1,0)$ (resp. $T(2,0)$). Also define $D_k = \emptyset$ for $k \leq 0$, and $D_{1/2}$ to be a square labeled 2. D'_m for $m \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ is obtained by rotating D_m by 180° and relabeling faces according to σ.

Remark 1. The definition of integer order diamond here differs from the one given by [3] by a reflection about a horizontal line.

Figure 4 shows the first few diamonds. Note that D'_m differs from D_m only by 1 or 2 squares.

Figure 5 shows D_3 and $D_{3+1/2}$ drawn in square blocks. Informally, for each integer n, D_n is the Aztec diamond of order n with squares replaced by blocks, such that the longest row is on s_0 and the rightmost block is [254], and $D_{n+1/2}$ resembles an n-by-$(n+1)$ Aztec diamond, with two extra squares.

For $m \in \frac{1}{2}\mathbb{Z}_{>0}$, [3] gives the number of perfect matchings of D_m:

$$|PM(D_m)| = \begin{cases} 2^{m(m+1)} & \text{if } m \in \mathbb{Z}_{>0} \\ 2^{(m+1/2)^2} & \text{if } m+1/2 \in \mathbb{Z}_{>0}. \end{cases}$$

Proposition 1. *Setting* $x_i = 1$,

$$y_N = \left| PM(D_{N/2}) \right|.$$

Remark 2. This can be easily checked by induction. The proof is omitted as it will follow from the more general theorem that will be proven later.

The proposition above makes us suspect that under a suitable weighting on the perfect matchings of $D_{N/2}$, y_N can be expressed as the weight of $D_{N/2}$, up to a monomial factor. By symmetry of Q under σ, y'_N should be related to $D'_{N/2}$.

We put weights on edges of graphs as follows: for an edge with two neighboring faces labeled a and b, weight it by $\frac{1}{x_a x_b}$.

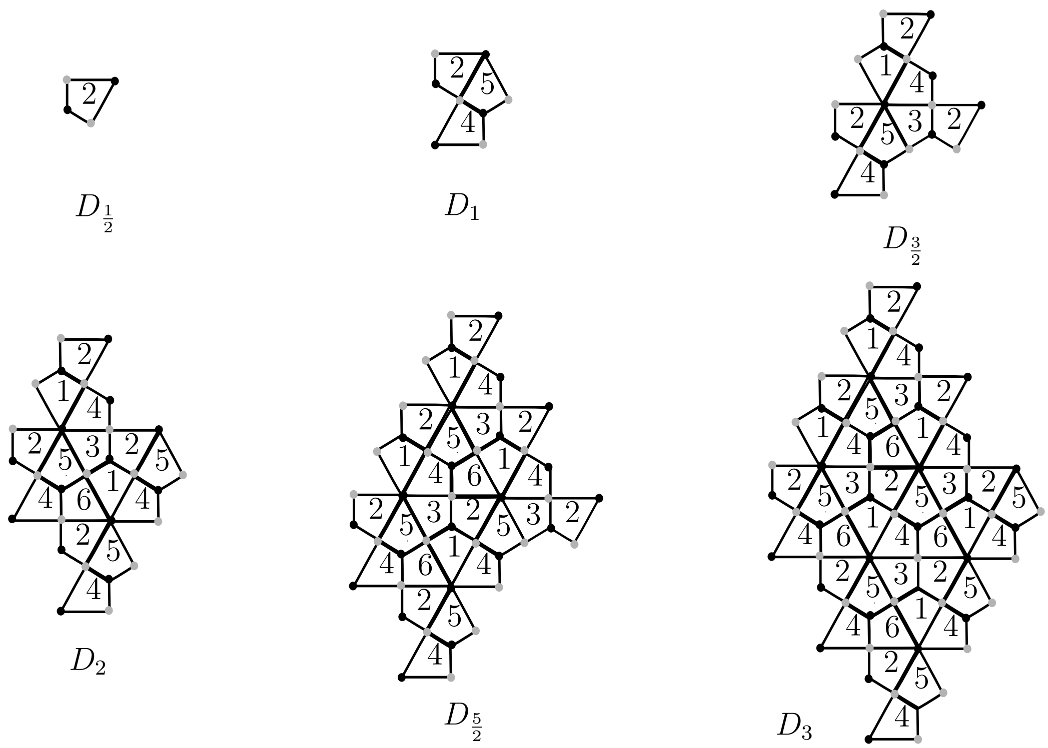

Figure 4: The first few diamonds, $D_{n/2}$ for $1 \leq n \leq 6$

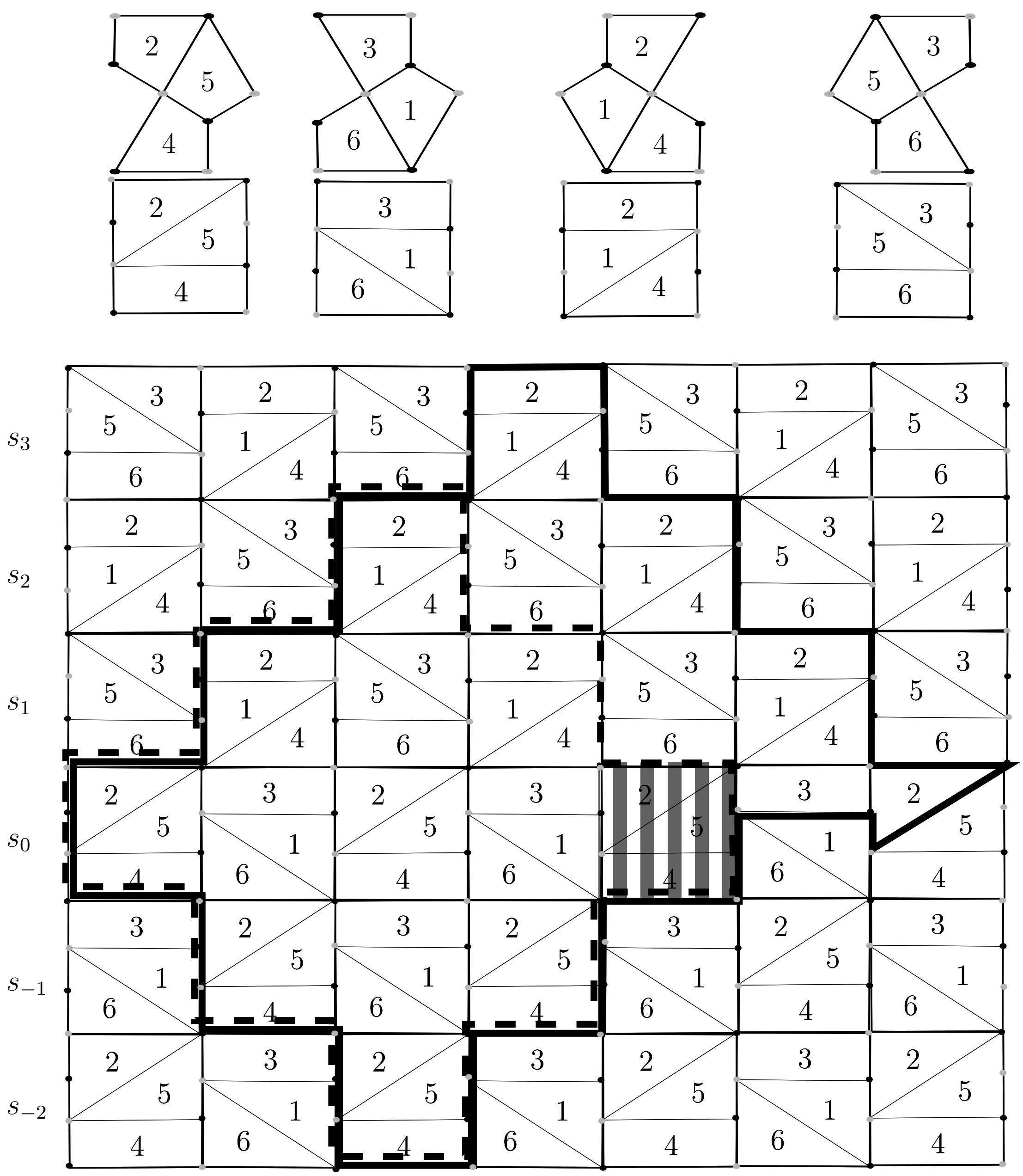

Figure 5: D_3 and $D_{3+1/2}$ drawn in square blocks. The shaded block is $T(0,0)$.

Definition. The *covering monomial*, $m(D)$, of a subgraph D of the brane tiling is a monomial in $\{x_i\}_{1\leq i\leq 6}$, where the exponent of x_i counts the number of faces of D and its neighboring faces in the brane tiling with the label i.

Remark 3. There is a more general definition of covering monomial in Jeong's paper [7].

Theorem. *For $N \in \mathbb{Z}_{>0}$,*

$$y_N = w(D_{N/2})m(D_{N/2}), \tag{2}$$

$$y'_N = w(D'_{N/2})m(D'_{N/2}). \tag{3}$$

The theorem follows from comparing the exchange relation with the two recursions on the weights and covering monomials of diamonds.

4 Recursion on the weights of diamonds

Proposition 2. *For $n \in \mathbb{Z}_{\geq 2}$,*

$$\begin{aligned} w(D_n)w(D_{n-3/2}) &= w(D_{n-1/2})w(D_{n-1})\frac{1}{x_1x_2}\frac{1}{x_3x_4}\frac{1}{x_5x_6} \\ &\quad + w(D'_{n-1/2})w(D'_{n-1})\frac{1}{x_1x_2}\frac{1}{x_2x_3}\frac{1}{x_3x_5}; \end{aligned} \tag{4}$$

For $n \in \mathbb{Z}_{\geq 1}$,

$$\begin{aligned} w(D_{n+1/2})w(D_{n-1}) &= w(D_n)w(D_{n-1/2})\frac{1}{x_1x_3}\frac{1}{x_2x_6}\frac{1}{x_4x_5} \\ &\quad + w(D'_n)w(D'_{n-1/2})\frac{1}{x_1x_3}\frac{1}{x_2x_3}\frac{1}{x_2x_5}. \end{aligned} \tag{5}$$

Proof. Let $m = n$ or $n + 1/2$ be a half-integer. We prove this by using graphical condensation, i.e. we need to show that a superposition of perfect matchings of D_m and $D_{m-3/2}$ can always be decomposed into a matching of $D_{m-1/2}$ and D_{m-1} or $D'_{m-1/2}$ and D'_{m-1}, with three extra edges, but not in both ways.

We will use Speyer's [13, page 37] formulation of Kuo's graphical condensation theorem [8] stated as follows. If a planar bipartite graph G has its vertices partitioned into

$$V(G) = \mathbf{N} \sqcup \mathbf{NE} \sqcup \mathbf{E} \sqcup \mathbf{SE} \sqcup \mathbf{S} \sqcup \mathbf{SW} \sqcup \mathbf{W} \sqcup \mathbf{NW} \sqcup \mathbf{C}$$

such that:

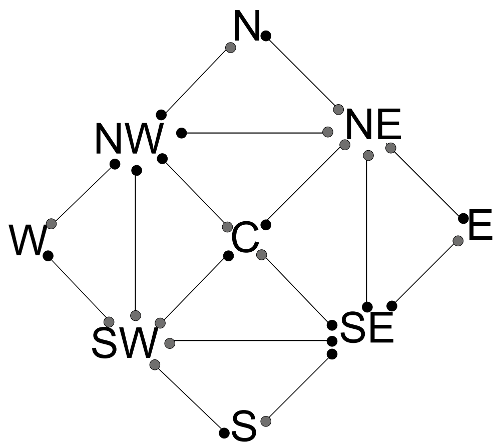

Figure 6: Possible edge connection among nine vertex components

1. Possible edge connection among the nine parts as described in Figure 6;
2. **NW**, **SE** each contains one more black vertex than white vertex, **NE**, **SW** each contains one more white vertex than black vertex, and **C**, **N**, **E**, **S**, **W** each contains the same number of vertices in either color;
3. Boundary vertices are all black for **NW** and **SE**, and all white for **NE**, **SW**.

Then

$$w(G)w(\mathbf{C}) = w(\mathbf{W}\cup\mathbf{NW}\cup\mathbf{SW}\cup\mathbf{C})w(\mathbf{E}\cup\mathbf{NE}\cup\mathbf{SE}\cup\mathbf{C})w(\mathbf{S})w(\mathbf{N})+ w(\mathbf{S}\cup\mathbf{SE}\cup\mathbf{SW}\cup\mathbf{C})w(\mathbf{N}\cup\mathbf{NE}\cup\mathbf{NW}\cup\mathbf{C})w(\mathbf{W})w(\mathbf{E}). \tag{6}$$

Now for $m = n \in \mathbb{Z}_{\geq 3}$, partition the vertices of D_n as shown in Figure 7. In fact $\mathbf{N} = \mathbf{E} = \emptyset$. It is easy to see that conditions 1 through 3 are all satisfied, so the method of graphical condensation can be applied, yielding (6). The LHS of (6) agrees with the LHS of (4) since $\mathbf{C} = D_{n-3/2}$. We can

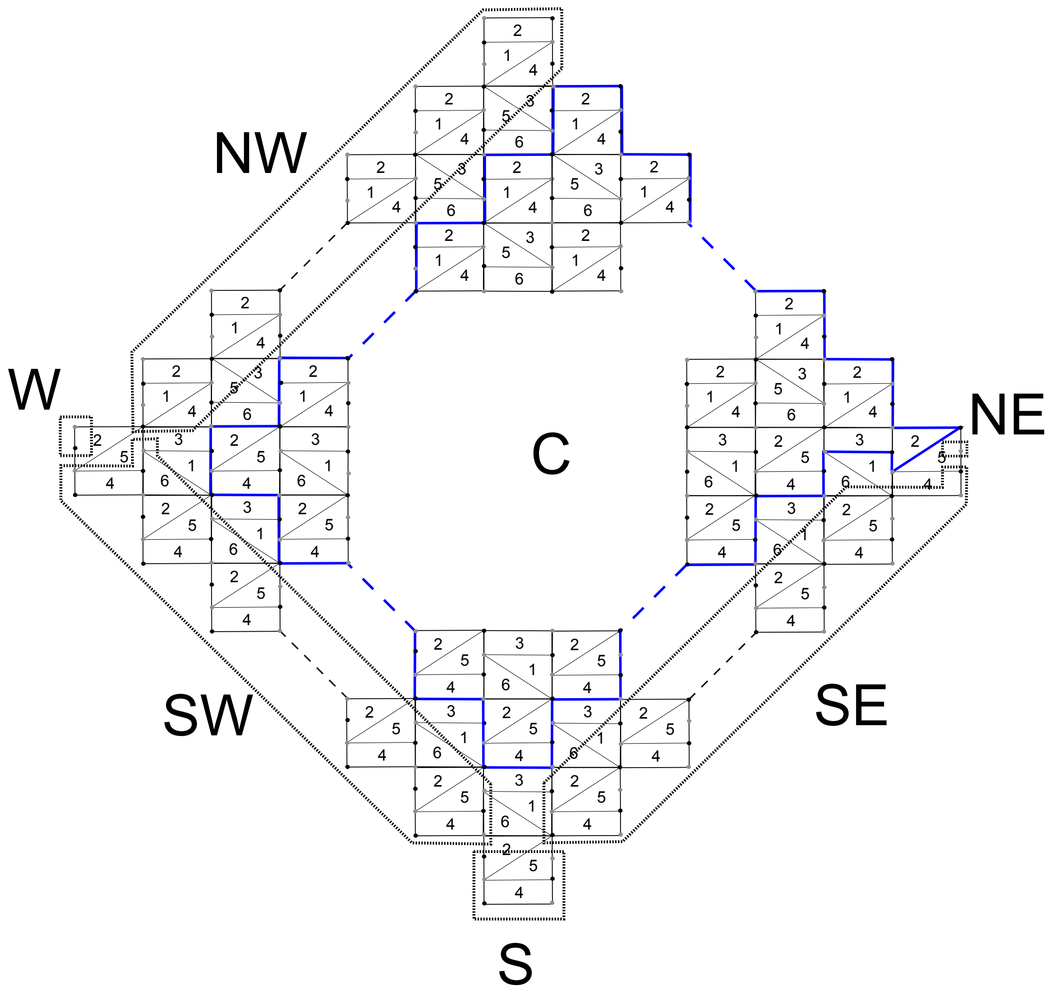

Figure 7: Partitioning the vertices of D_n into 7 parts

see from Figure 8 that the RHS of (6) also agrees with the RHS of (4):

$$\begin{aligned}
\mathbf{W} \cup \mathbf{NW} \cup \mathbf{SW} \cup \mathbf{C} &= D_{n-1/2}, \\
\mathbf{E} \cup \mathbf{NE} \cup \mathbf{SE} \cup \mathbf{C} &= D_{n-1}, \\
w(\mathbf{S}) &= \frac{1}{x_1x_2}\frac{1}{x_3x_4}\frac{1}{x_5x_6}; \\
\mathbf{S} \cup \mathbf{SE} \cup \mathbf{SW} \cup \mathbf{C} &= D'_{n-1/2}, \\
w(\mathbf{N} \cup \mathbf{NE} \cup \mathbf{NW} \cup \mathbf{C}) &= w(D'_{n-1})\frac{1}{x_1x_2}\frac{1}{x_3x_5}, \\
w(\mathbf{W}) &= \frac{1}{x_2x_3}.
\end{aligned}$$

Note that although $\mathbf{N} \cup \mathbf{NE} \cup \mathbf{NW} \cup \mathbf{C} \neq D'_{n-1}$, its perfect matchings must contain two of the edges (colored in red), with weights $\frac{1}{x_1x_2}$ and $\frac{1}{x_3x_5}$, leaving a perfect matching of D'_{n-1}. Also, $w(\mathbf{N}) = w(\mathbf{E}) = w(\emptyset) = 1$.

The case $n = 2$ involves the diamond $D_{1/2}$ which is defined separately to be a single square labeled 2. In fact the vertices of D_2 can still be partitioned as shown in Figure 7.

Equation (5) is similarly proven by applying graphical condensation; see Figure 9 and Figure 10 for $n \geq 2$. In fact Figure 9 is valid even when $\mathbf{C} = \emptyset = D_0$, so (5) also holds for $n = 1$.

□

5 Recursion on the covering monomials of diamonds

Proposition 3. *For $n \in \mathbb{Z}_{\geq 2}$,*

$$\begin{aligned}
m(D_n)m(D_{n-3/2}) &= m(D_{n-1/2})m(D_{n-1})(x_1x_2)(x_3x_4)(x_5x_6) \\
&= m(D'_{n-1/2})m(D'_{n-1})(x_1x_2)(x_2x_3)(x_3x_5); \qquad (7)
\end{aligned}$$

$$\begin{aligned}
m(D_{n+1/2})m(D_{n-1}) &= m(D_n)m(D_{n-1/2})(x_1x_3)(x_2x_6)(x_4x_5) \\
&= m(D'_n)m(D'_{n-1/2})(x_1x_3)(x_2x_3)(x_2x_5). \qquad (8)
\end{aligned}$$

Proof. We will count the squares and compute both sides. Let $n \in \mathbb{Z}_{\geq 1}$, define $f_n \in \mathbb{N}^6$ where $(f_n)_i$ is the number of squares labeled i in D_n.

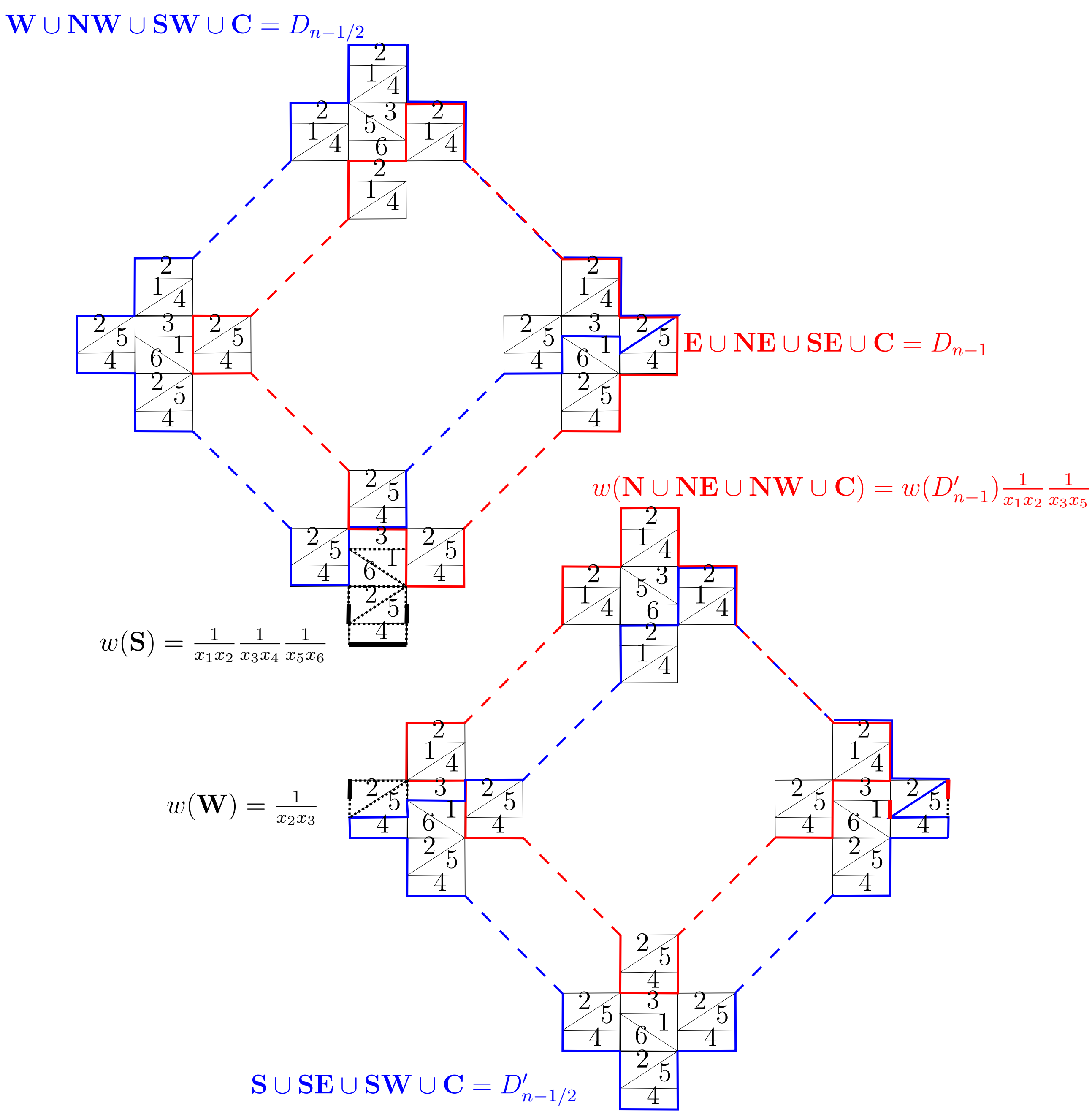

Figure 8: D_n plus $D_{n-3/2}$ decomposed into two smaller diamonds, in two ways

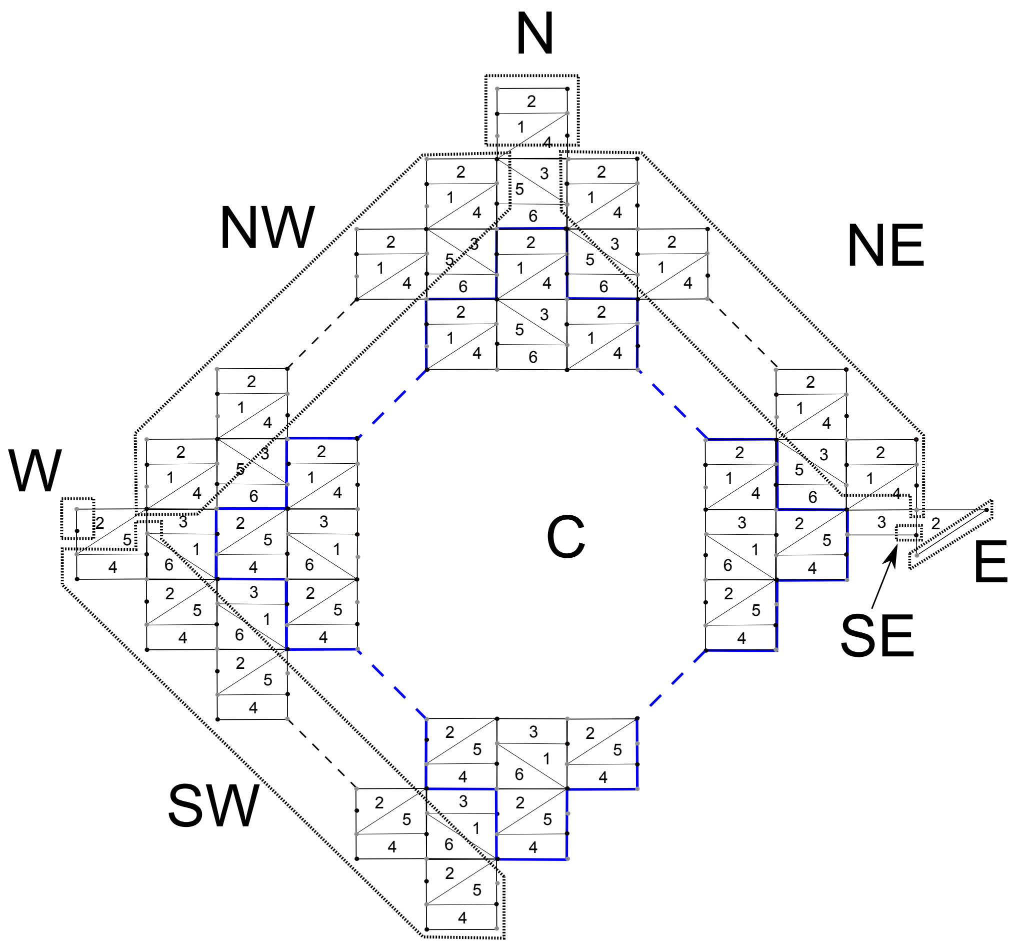

Figure 9: Partitioning the vertices of $D_{n+1/2}$ into 7 parts

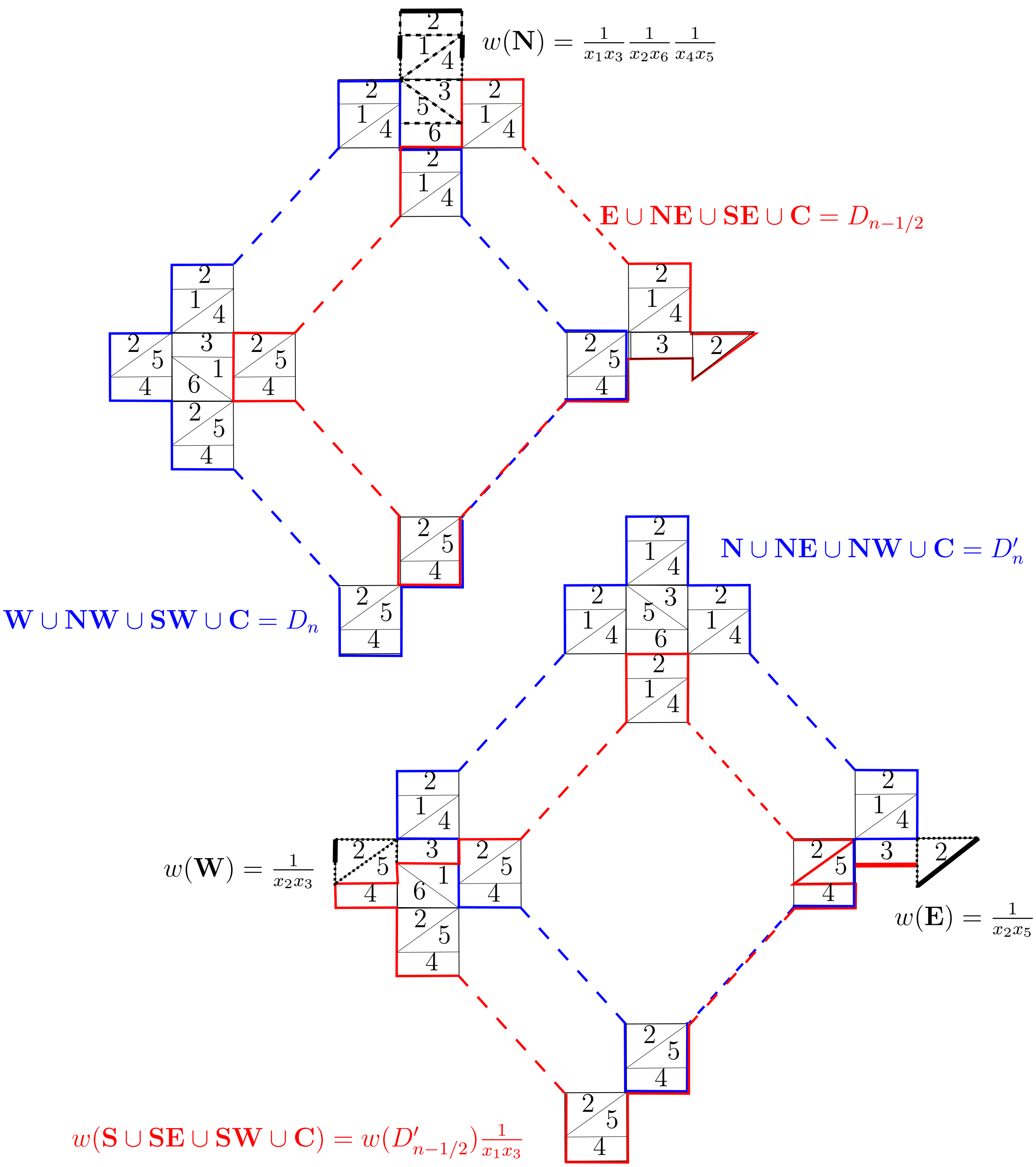

Figure 10: $D_{n+1/2}$ plus D_{n-1} decomposed into two smaller diamonds, in two ways

The number of blocks of [254] in D_n is

$$1 + 2 + \ldots + n = \frac{1}{2}n(n+1).$$

Similarly the number of blocks of [316], [214], [356] is respectively $\frac{1}{2}n(n-1), \frac{1}{2}n(n-1), \frac{1}{2}(n-1)(n-2)$.

Using multi-index notation,

$$\begin{aligned}\mathbf{x}^{f_n} &= (x_2x_5x_4)^{n(n+1)/2}(x_3x_1x_6)^{n(n-1)/2}(x_2x_1x_4)^{n(n-1)/2}(x_3x_5x_6)^{(n-1)(n-2)/2} \\ &= \mathbf{x}^{(n(n-1),n^2,(n-1)^2,n^2,n(n-1)+1,(n-1)^2)},\end{aligned}$$

i.e.

$$f_n = (n(n-1), n^2, (n-1)^2, n^2, n(n-1)+1, (n-1)^2).$$

Similarly

$$f_{n+1/2} = (n^2, n(n+1)+1, n(n-1)+1, n(n+1), n^2, n(n-1)).$$

Let $(h_n)_i$ be the number of neighboring squares of D_n labeled i, then

$$\begin{aligned}\mathbf{x}^{h_n} &= x_6\left[(x_3x_6)(x_5x_6)\right]^{n-1}\left[(x_1x_3)(x_3x_6)\right]^{n} x_3 \\ &= \mathbf{x}^{(n,0,3n,0,n-1,3n-1)},\end{aligned}$$

and

$$\begin{aligned}\mathbf{x}^{h_{n+1/2}} &= x_6\left[(x_3x_6)(x_5x_6)\right]^{n}\left[(x_1x_3)(x_3x_6)\right]^{n-1}(x_1x_3)(x_1x_6x_5) \\ &= \mathbf{x}^{(n+1,0,3n,0,n+1,3n+1)}.\end{aligned}$$

So,

$$\begin{aligned}m(D_n) &= \mathbf{x}^{(f_n+h_n)} \\ &= \mathbf{x}^{(n(n-1),n^2,(n-1)^2,n^2,n(n-1)+1,(n-1)^2)+(n,0,3n,0,n-1,3n-1)} \\ &= \mathbf{x}^{(n^2,n^2,n^2+n+1,n^2,n^2,n^2+n)},\end{aligned} \tag{9}$$

$$\begin{aligned}m(D_{n+1/2}) &= \mathbf{x}^{(f_{n+1/2}+h_{n+1/2})} \\ &= \mathbf{x}^{(n^2,n(n+1)+1,n(n-1)+1,n(n+1)+(n+1,0,3n,0,n+1,3n+1)} \\ &= \mathbf{x}^{(n^2+n+1,n^2+n+1,n^2+2n+1,n^2+n,n^2+n+1,n^2+2n+1)},\end{aligned} \tag{10}$$

and $m(D'_m) = \sigma(m(D_m))$ (for any positive half-integer m) by permuting entries in the exponent. Note that although f and h above give the wrong count for the special case $D_{1/2}$, (10) for $n = 0$ still gives the correct expression for $m(D_{1/2}) = x_1x_2x_3x_5x_6$. Also, formally define $m(D_0)$ by (9): $m(D_0) = x_3$.

So, for all $n \in \mathbb{Z}_{\geq 2}$ the expressions in (7) all evaluate to

$$\mathbf{x}^{(2n^2-3n+3,2n^2-3n+3,2n^2-n+2,2n^2-3n+2,2n^2-3n+3,2n^2-n+1)},$$

and for all $n \in \mathbb{Z}_{\geq 1}$ the expressions in (8) all evaluate to

$$\mathbf{x}^{(2n^2-n+2,2n^2-n+2,2n^2+n+2,2n^2-n+1,2n^2-n+2,2n^2+n+1)}.$$

In fact, it is easier to see the proposition is true without explicit computations, but by simply looking at the "cover" of diamonds (i.e. faces plus neighboring faces) and comparing their overlaps.

In the upper half of Figure 11, the "covers" of $D_{n-1/2}$ (colored blue) and D_{n-1} (red) intersect at a purple area, which is exactly the "cover" of $D_{n-3/2}$, while their union together with 6 extra faces (shaded) is exactly the "cover" of D_n, proving the first equality of (7). The lower half of Figure 11 proves the second equality of (7), and Figure 12 proves (8).

Note that the 6 extra faces (shaded) in Figure 11 and Figure 12 are paired up by the 3 extra edges (thickened), which combines to cancel out the extra edge weight in (4) and (5).

□

6 Proof of the theorem

Proof. Let $N \geq 3$ and assume $y_k = w(D_{k/2})m(D_{k/2}), y'_k = w(D'_{k/2})m(D'_{k/2})$ for all $k < N$. Write $z_N = w(D_{N/2})m(D_{N/2})$. Apply (4) and (7) if N is odd, or (5) and (8) if N is even:

$$\begin{aligned} z_N y_{N-3} &= w(D_{N/2})m(D_{N/2})w(D_{N/2-3/2})m(D_{N/2-3/2}) \\ &= w(D_{N/2-1/2})w(D_{N/2-1})m(D_{N/2-1/2})m(D_{N/2-1}) \\ &\quad + w(D'_{N/2-1/2})w(D'_{N/2-1})m(D'_{N/2-1/2})m(D'_{N/2-1}) \\ &= y_{N-1}y_{N-2} + y'_{N-1}y'_{N-2} \\ &= y_N y_{N-3} \end{aligned}$$

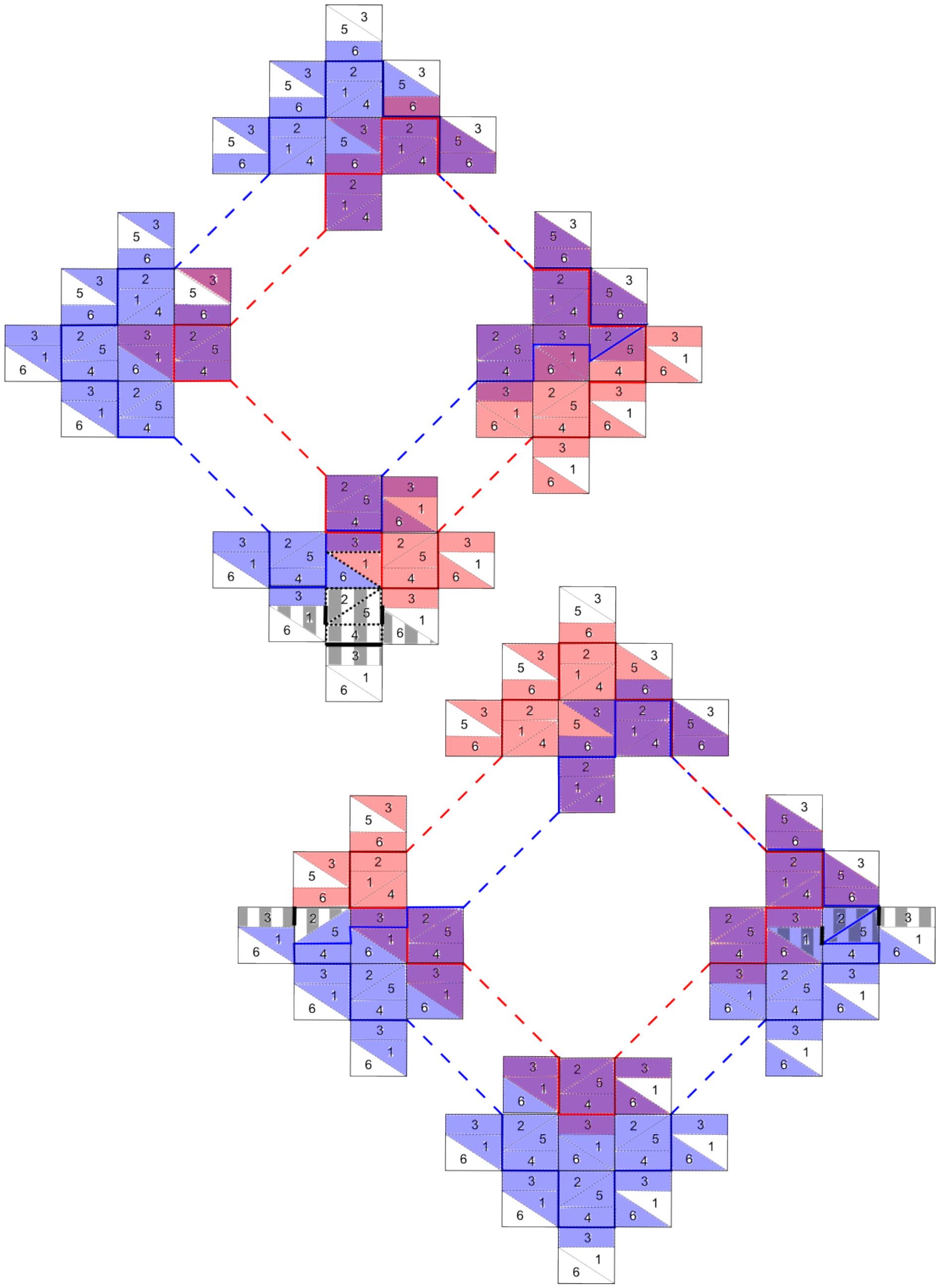

Figure 11: "cover" of D_m plus $D_{m-3/2}$ decomposed into "cover" of two smaller diamonds, in two ways

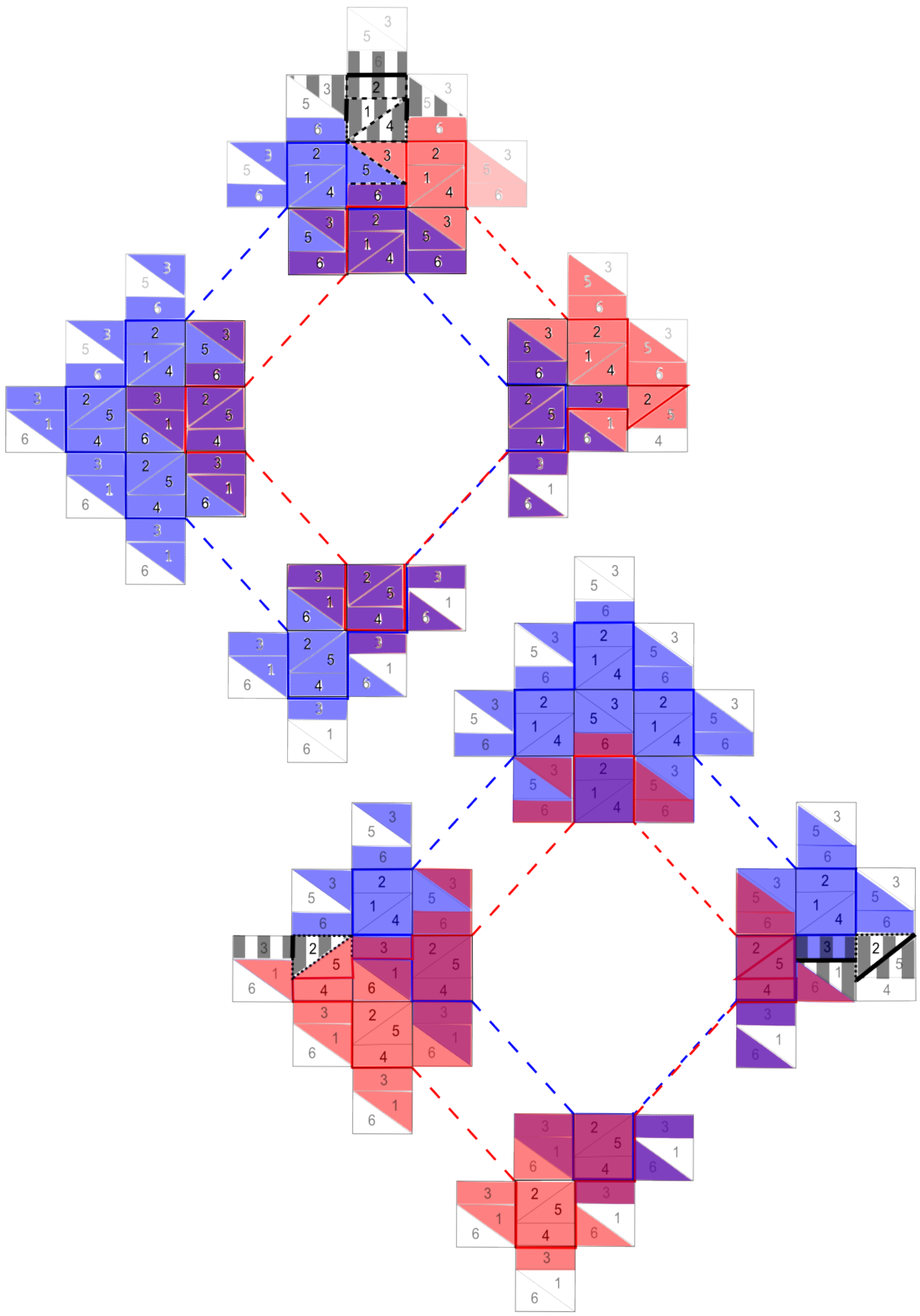

Figure 12: "cover" of $D_{m+1/2}$ plus D_{m-1} decomposed into "cover" of two smaller diamonds, in two ways

agrees with exchange relation (1), so $z_N = w(D_{N/2})m(D_{N/2}) = y_N$. Act by σ on both sides: $w(D'_{N/2})m(D'_{N/2}) = y'_N$. The base case of $N = 0$ becomes: $y_0 = 1 \cdot x_3 = w(D_0)m(D_0)$, $y'_0 = 1 \cdot x_6 = w(D'_0)m(D'_0)$ (not interesting, but it is formally consistent). The base cases of $N = 1, 2$ can be checked directly, or are tree phenomena as discussed by Jeong [7]. □

7 Acknowledgments

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