STABLE CLUSTER VARIABLES

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Abstract. Eager and Franco introduced a transformation on the F-polynomials of Fomin and Zelevinsky that seems to display a surprising stabilization property in the case of the dP1 quiver. They conjectured that this transformation will always cause the cluster variables to converge as a formal power series. We explore this transformation on the Kronecker and Conifold quivers and show that cluster variable stabilization occurs for both of them. We also provide a combinatorial interpretation for the stable terms in each case.

Contents

1. Cluster Algebras and F-Polynomials

Fomin and Zelevinsky defined cluster algebras and quiver mutations in [1]. They also introduced F-polynomials in [2]. We review these concepts here very briefly.

Definitions 1.1.

A **quiver** is a directed graph. Multiple edges are allowed. Self-loops are not allowed.

Given a quiver, a **mutation sequence** is an infinite sequence of vertices in the quiver.

Given a quiver, a **cluster seed** is an assignment of values, one to each vertex.

Given a quiver, mutation sequence, and cluster seed, **quiver mutation/ cluster mutation** is performed by reading the mutation sequence left to right, and for each vertex i in the sequence, mutating the quiver at i according to the mutation rules given below. Quiver mutation generates an infinite sequence of new quivers along with an infinite sequence of new cluster variables. For an example of quiver mutation, see Figure 2

Definition 1.2. Rules for mutation at a vertex i:

(1) Update the cluster variable for vertex i:

$$\frac{\prod_{v \to i} \text{cluster var for } v + \prod_{i \to v} \text{cluster var for } v}{\text{old cluster var for } i}$$

(2) For every 2-path $u \to i \to v$, draw an arrow $u \to v$.
(3) If any self-loops or 2-cycles were newly created, delete them.
(4) Reverse all arrows incident to i.

Definition 1.3. Given a quiver, we **frame** it by adding a new "frozen vertex" i' for each vertex i and drawing an arrow $i \to i'$.

Definition 1.4. Given a framed quiver and a mutation sequence μ that includes only non-frozen vertices, set the initial cluster variable corresponding to any non-frozen vertex equal to 1 and the initial cluster variable corresponding to any frozen vertex i' equal to y_i. The cluster variables obtained from μ are known as **F-polynomials**.

2. Stable Cluster Variables

We now summarize the apparently stabilizing transformation on F-polynomials introduced by Eager and Franco.

Definition 2.1. At any step in the mutation sequence, define the **C-matrix** by

$$C_{ij} = \text{number of arrows from } i' \text{ to } j$$

(with this value being negative if the arrows are pointing from j to i')

Definition 2.2. Given a C-matrix and some monomial $m = y_1^{a_1} \cdot y_2^{a_2} \cdot \ldots \cdot y_n^{a_n}$, define the **C-matrix transformation** $\tilde{m}$ of m as

$$\tilde{m} = y_1^{b_1} \cdot y_2^{b_2} \cdot \ldots \cdot y_n^{b_n}$$
$$\text{where } \vec{b} = C^{-1}\vec{a}$$

Definition 2.3. Let F_n be the F-polynomial derived at the nth step of the mutation sequence. Let C_n be the C-matrix at the nth step of the mutation sequence. Writing $F_n = \sum m$ as a sum of monomials m, define the **C-matrix transformation** $\tilde{F}_n$ of F_n as

$$\tilde{F}_n = \sum \tilde{m}$$

In other words, we transform each monomial individually.

In section 9.5 of [3] Eager and Franco observe that for the dP1 quiver, the sequence of transformed cluster variables $\tilde{F}_n$ displays a surprising stabilization property, appearing to converge to a limit as a formal power series. They conjecture that the transformed cluster variables will always converge for any quiver.

In the remainder of the paper, we present two examples of quivers where this convergence holds. We also interpret the transformed F-polynomials and limit combinatorially. In the context of these examples, we will refer to the transformed F-polynomials $\tilde{F}_n$ as **stable cluster variables**.

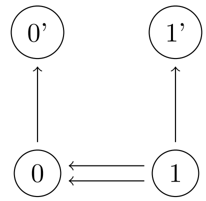


FIGURE 1. Framed Kronecker Quiver

3. KRONECKER

The first example we present is the Kronecker quiver, pictured in Figure 1, with mutation sequence $(0, 1, 0, 1, \ldots)$.

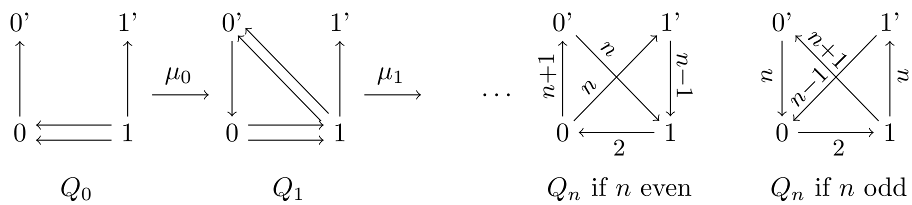


FIGURE 2. The Kronecker quiver mutates with a predictable structure.

n	F_n	$\tilde{F}_n$
1	$y_0 + 1$	$y_0 + 1$
2	$y_0^2y_1 + y_0^2 + 2y_0 + 1$	$y_0^2y_1^4 + 2y_0y_1^2 + y_1 + 1$
3	$y_0^2y_1^2 + 2y_0^3y_1 + y_0^3 + 2y_0^2y_1 + 3y_0^2 + 3y_0 + 1$	$y_0^9y_1^6 + 3y_0^6y_1^4 + 2y_0^5y_1^3 + 3y_0^3y_1^2 + 2y_0^2y_1 + y_0 + 1$
4	$\ldots + 6y_0^2y_1 + 4y_0^3 + 3y_0^2y_1 + 6y_0^2 + 4y_0 + 1$	$\ldots + 3y_0^4y_1^6 + 4y_0^3y_1^4 + 3y_0^2y_1^3 + 2y_0y_1^2 + y_1 + 1$

FIGURE 3. Table of the first few cluster variables, illustrating the stabilization property. The low order terms of the stable cluster variables match, up to a fluctuation between y_0 and y_1. (Entries in the last row are truncated).

Lemma 3.1. *The following recurrence on F-polynomials holds:*

$$F_0 = 1$$

$$F_1 = y_0 + 1$$

$$F_n F_{n-2} = y_0^n y_1^{n-1} + F_{n-1}^2 \text{ for } n \geq 2$$

Proof. Q_n has a predictable structure, as shown in Figure 2. Using the rule for updating cluster variables (part 1 of Definition 1.2), it is easy to read off this recurrence from the structure of Q_n. □

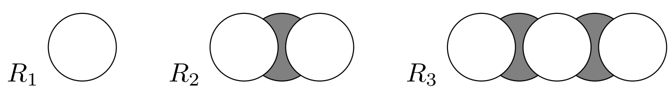
Lemma 3.2.

$$C_n = \begin{cases} \begin{bmatrix} -(n+1) & n \\ -n & n-1 \end{bmatrix} & \text{if } n \text{ even} \\ \\ \begin{bmatrix} n & -(n+1) \\ n-1 & -n \end{bmatrix} & \text{if } n \text{ odd} \end{cases} \qquad C_n^{-1} = \begin{cases} \begin{bmatrix} n-1 & -n \\ n & -(n+1) \end{bmatrix} & \text{if } n \text{ even} \\ \\ \begin{bmatrix} n & -(n+1) \\ n-1 & -n \end{bmatrix} & \text{if } n \text{ odd} \end{cases}$$

Proof. Q_n has a predictable structure, as shown in Figure 2. Using the definition of the C-matrix, it is easy to read it off from the structure of Q_n. ☐

The two possible forms of C_n^{-1} can be obtained from each other by switching the rows. This minor discrepancy accounts for the fluctuation between variables seen in Figure 3. We will from now on remove this fluctuation by eliminating one case, in order to simplify computation. We choose to assume all cases follow the case when n is odd.

Definition 3.3. Let the **row pyramid** of length n, R_n, be the two-layer arrangement of stones with n white stones on the top layer and $n-1$ black stones on the bottom layer, as shown below.



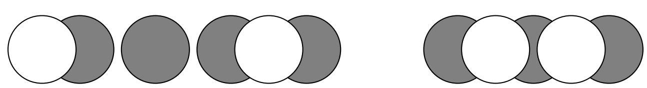
Definitions 3.4.

A **partition** of R_n is a stable configuration achieved by removing stones from R_n. That is, if a stone is removed, any stone above it must be removed. (We will draw partitions by showing the non-removed stones).

For any partition P of R_n, its **weight**

$$weight(P) = y_0^{\#\text{ white stones removed}} y_1^{\#\text{ black stones removed}}$$

Example 3.5. A partition of R_9 with weight $y_0^5 y_1$.



Proposition 3.6. *F_n is the following generating function / partition function for R_n.*

$$F_n = \sum_{Partitions\ P\ of\ R_n} weight\ (P)$$

Proof. Omitted, but follows from the inductive structure of F_n. ☐

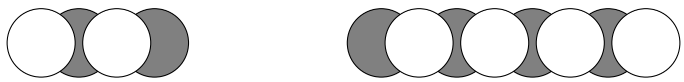
Proposition 3.7.

$$\tilde{F}_n = \sum_{Partitions\ of\ R_n} \frac{(y_0^n y_1^{n-1})^{\ \#\ white\ stones\ removed}}{(y_0^{n+1} y_1^n)^{\#\ black\ stones\ removed}}$$

Proof. Since we know the form of the C_n-matrix, we have that for any monomial $m = y_0^a, y_1^b$, it transforms to $\tilde{m} = y_0^{n(a-b)-b}, y_1^{n(a-b)-a}$. Transform the expression in the previous proposition according to this rule and regroup terms. □

Definition 3.8. A **simple partition** of R_n is a partition of R_n such that the removed white stones form one consecutive block, and no exposed black stones remain. (With possibly no stones removed).

Example 3.9. A simple partition of R_9.



The idea of the proof of the next theorem is that the stable terms in $\tilde{F}_n$ are the contributions exactly from the simple partitions.

Theorem 3.10. *For the Kronecker quiver with* $\mu = (0, 1, 0, 1, \ldots)$

$$\lim_{n\to\infty} \tilde{F}_n = 1 + y_0 + 2y_0^2 y_1 + 3y_0^3 y_1^2 + 4y_0^4 y_1^3 + \ldots$$

Proof. The term 1 clearly stabilizes, since every F-polynomial includes 1 as a term (coming from the partition with no stones removed), and the C-matrix transforms it to 1.

Claim: For any monomial $y_0^a y_1^b \neq 1$ in F_n, $a > b$.
In R_n it is impossible to remove as many black stones as white stones, since a black stone can only be removed after both white stones on top of it have been removed. Since F_n is the partition function for R_n the claim follows.

Claim: For any monomial $y_0^{a'} y_1^{b'} \neq 1$ in $\tilde{F}_n$, $a' > b'$.
Let $m = y_0^a y_1^b$ be any monomial in F_n. The C_n matrix transforms it to $\tilde{m} = y_0^{n(a-b)-b} y_1^{n(a-b)-a}$. By the previous claim, $b < a$. The claim follows.

So let $\tilde{m} = y_0^a y_1^{a-k}$ be a monomial, with $k \geq 1$.

Case 1: $k = 1$. So $\tilde{m} = y_0^a y_1^{a-1}$. We claim that there is some N such that for all $n \geq N$, $\tilde{m}$ appears in $\tilde{F}_n$ with coefficient a.

Using the C_n-matrix, $\tilde{m}$ appears in $\tilde{F}_n$ if and only if the term $y_0^{n-a+1} y_1^{n-a}$ appears in F_n. This term corresponds to a partition with $(n-a+1)$ white stones removed and $(n-a)$ black stones removed. Note that since the difference is 1, it must be a simple partition. It is an easy combinatorial observation to see that there are a such partitions whenever $n \geq a$ and 0 such partitions whenever $n < a$. So the claim holds with $N = a$.

Case 2: $k \geq 2$. We claim that for sufficiently large n, $\tilde{m} = y_0^a y_1^{a-k}$ does not appear in $\tilde{F}_n$.

Suppose $\tilde{m}$ appears in $\tilde{F}_z$ for some z. Using the matrix C_z, it corresponds to the term $y_0^{zk-a+k} y_1^{zk-a}$ in F_z. If $\tilde{m}$ appears in $\tilde{F}_{z+1}$, then it corresponds to the term $y_0^{zk-a+2k} y_1^{zk-a+k}$ in F_{z+1}, using C_{z+1}. That is, we add k to each exponent. However, increasing from z to $z+1$ adds only one stone of each color to R_z. So if $k \geq 2$, then after a finite number of steps, the exponents will grow too large for any possible partition. □

We now give a combinatorial interpretation for $\lim_{n\to\infty} \tilde{F}_n$. This interpretation will generalize in the next example presented in this paper.

Definition 3.11. Let R_∞ be the row pyramid extending infinitely toward the center as shown.

Definitions 3.12.

A **partition** of R_∞ is a stable configuration achieved by removing **an infinite number of stones, such that only a finite number of stones remains**.

A **simple partition** of R_∞ is a partition of R_∞ such that the removed white stones form one consecutive (infinite) block, and no exposed black stones remain.

Define the **weight** of a partition P of R_∞ as

$$weight(P) = y_0^{\#\text{ non-removed white stones } + 1} y_1^{\#\text{ non-removed black stones}}$$

Note that the number of non-removed white/black stones is actually the same. We have written the expression in this form simply to make it look more similar to a familiar weight function.

Example 3.13. A simple partition of R_∞ with weight $y_0^4 y_1^3$.

Definition 3.14. Define a partition function

$$S = \sum_{\text{Simple partitions } P \text{ of } R_\infty} weight(P)$$

Proposition 3.15.

$$\lim_{n\to\infty} \tilde{F}_n = 1 + S$$

4. Conifold

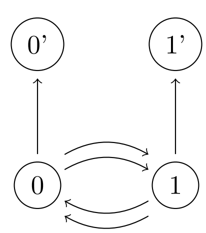


Figure 4. Framed Conifold Quiver

The second example we present is the Conifold quiver, pictured in Figure 4, with mutation sequence $(0, 1, 0, 1, \ldots)$. Note that the conifold is a quiver with length-2 cycles, which according to some conventions we cannot mutate. For this example, we will just mutate the quiver as usual, but at every step, remove any self-loops that were created.

A table again suggests that the C-matrix transformation stabilizes the cluster variables. (Entries in the last two rows are truncated).

n	F_n	$\tilde{F}_n$
1	$y_0 + 1$	$\underline{y_0 + 1}$
2	$y_0^4y_1 + 2y_0^3y_1 + y_0^2y_1 + y_0^2 + 2y_0 + 1$	$y_0^2y_1^5 + y_0^2y_1^4 + 2y_0y_1^3 + 2y_0y_1^2 + \underline{y_1 + 1}$
3	$\ldots + 6y_0^3y_1 + y_0^3 + 2y_0^2y_1 + 3y_0^2 + 3y_0 + 1$	$\ldots + 4y_0^4y_1^2 + 3y_0^3y_1^2 + \underline{2y_0^3y_1 + 2y_0^2y_1 + \underline{y_0 + 1}}$
4	$\ldots + 12y_0^3y_1 + 4y_0^3 + 3y_0^2y_1 + 6y_0^2 + 4y_0 + 1$	$\ldots + \underline{4y_0^2y_1^4 + 3y_0^2y_1^3 + \underline{2y_0y_1^3 + 2y_0y_1^2 + \underline{y_1 + 1}}}$

Here is a larger number of stable terms:

$$\ldots + 33y_0^{10}y_1^6 + 60y_0^9y_1^7 + 63y_0^9y_1^6 + 8y_0^8y_1^7 + 10y_0^9y_1^5 + 40y_0^8y_1^6 + 32y_0^8y_1^5 + 7y_0^7y_1^6 + 3y_0^8y_1^4 + 28y_0^7y_1^5 + 14y_0^7y_1^4 + 6y_0^6y_1^5 + 16y_0^6y_1^4 + 6y_0^6y_1^3 + 5y_0^5y_1^4 + 10y_0^5y_1^3 + y_0^5y_1^2 + 4y_0^4y_1^3 + 4y_0^4y_1^2 + 3y_0^3y_1^2 + 2y_0^3y_1 + 2y_0^2y_1 + y_0 + 1$$

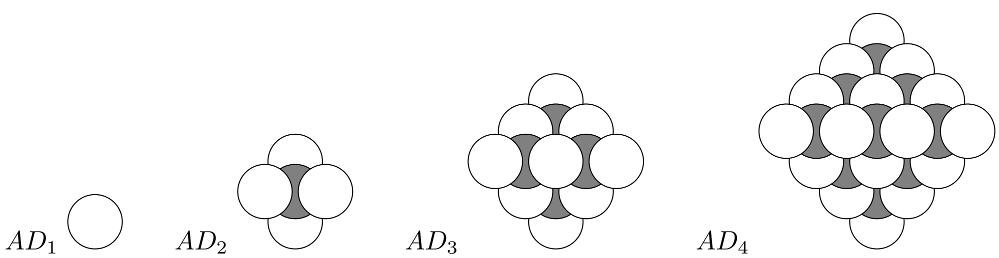
Whatever pattern these terms follow does not seem so easy to discern.

The conifold also mutates with a predictable structure, and it is easy to see that the C-matrix has the same form as in Section 3 with the Kronecker quiver. As we did in Section 3, we will eliminate the even case to remove the fluctuation in variables in $\tilde{F}_n$:

Lemma 4.1. $C_n = C_n^{-1} = \begin{bmatrix} n & -(n+1) \\ n-1 & -n \end{bmatrix}$

For the conifold quiver the stable cluster variables converge, and the limit can be combinatorially interpreted in an analogous way as in the case of the Kronecker quiver. Before we show this, we introduce some definitions.

Definition 4.2. Let AD_n be the 2-color Aztec diamond pyramid with n white stones on the top layer, as shown below.



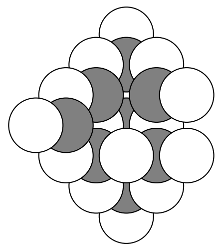
Definitions 4.3.

As in the previous section, a **partition** of AD_n is a stable configuration achieved by removing stones from AD_n.

As in the previous section, for any partition P of AD_n, its **weight**

$$weight(P) = y_0^{\#\text{ white stones removed}} y_1^{\#\text{ black stones removed}}$$

Example 4.4. A partition of AD_4 with weight $y_0^4 y_1^2$

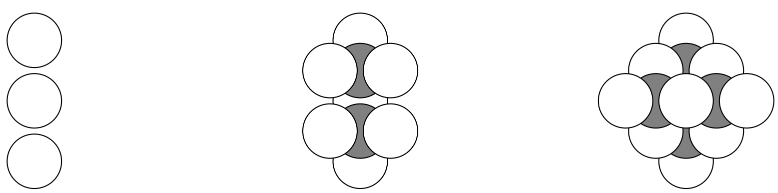


Theorem 4.5. *The F-polynomials are partition functions of AD_n.*

$$F_n = \sum_{Partitions\ P\ of\ AD_n} weight(P)$$

Proof. Proven in [4] by Elkies-Kuperberg-Larsen-Propp, but using a graph perfect matching instead of stone pyramid interpretation. (The two interpretations are equivalent). ☐

Note that each AD_n can be decomposed into layers of row pyramids (Definition 3.3), such that the kth layer from the top contains k row pyramids of length $n-k+1$. We will frequently refer to a row pyramid in the decomposition simply as a **row** of AD_n.



(A) 3 rows of length 1 (B) 2 rows of length 2 (C) 1 row of length 3

Figure 5. Row pyramid decomposition of AD_3, shown layer by layer.

Definitions 4.6.

A **simple partition** of AD_n is a partition such that the restriction of the partition to each row pyramid is simple.

For any partition P of AD_n, for any row r of AD_n, we call r **altered** if at least one stone is removed from r.

Example 4.4 above is a simple partition with 2 altered rows.

Analogous to the situation in Section 3, the idea of the next proof is that the stable terms in $\tilde{F}_n$ are contributed by the simple partitions.

Theorem 4.7. *For the conifold,* $\lim_{n\to\infty} \tilde{F}_n$ *converges as a formal power series.*

Proof. The term 1 is clearly in the limit, since each F_n includes it, and C_n transforms it to itself. For the same reason as in Section 3, for every monomial $\tilde{m} = y_0^a y_1^b \neq 1$, if $\tilde{m}$ appears in $\tilde{F}_n$ for any n, then $a > b$.

Claim: Let $\tilde{m} = y_0^a y_1^{a-k}$, with $k \geq 1$. For sufficiently large n, the terms in F_n transforming to $\tilde{m}$ come only from simple partitions (possibly none).

Proof:
Suppose there is a z such that there is a partition P of AD_z with weight m_1 transforming to $\tilde{m}$. Then it must be that $m_1 = y_0^{zk-a+k} y_1^{zk-a}$. In P, k is the difference between the number of white stones and black stones removed. Let S be the set of rows altered by P. Note that P is a simple partition iff $|S| = k$, and P is a non-simple partition iff $|S| < k$.

Suppose $|S| < k$.

If F_{z+1} has a term transforming to $\tilde{m}$, it must be $m_2 = y_0^{zk-a+2k} y_1^{zk-a+k}$. In other words, each exponent increases by k from m_1. But increasing from z to $z+1$ adds only one stone of each color to each row. So if $k > |S|$, then after a finite number of steps it will be impossible for any partition **altering exactly the rows in S** to have a weight transforming to $\tilde{m}$. Since this is true for any set S of fewer than k rows, eventually the only possible partitions with weight transforming to $\tilde{m}$ will be simple partitions.

This proof easily be can be modified to show that the following stronger claim holds: For sufficiently large n, the terms in F_n transforming to $\tilde{m}$ come only from simple partitions, such that each altered row of the partition has more than y stones removed, for any fixed y.

Claim: For sufficiently large n, the coefficient in front of $\tilde{m}$ in $\tilde{F}_n$ is constant.

Proof:
Assume n is large enough that all partitions with weight transforming to $\tilde{m}$ are simple with each altered row having strictly greater than 1 stone removed, and that $n \geq k$. The second condition guarantees that AD_n is large enough for every possible set of k altered rows to exist.

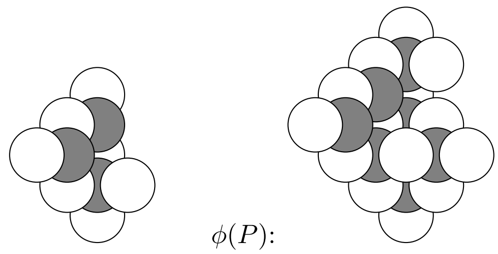
We construct a bijection ϕ between partitions of AD_n with weight transforming to $\tilde{m}$ and partitions of AD_{n+1} with weight transforming to $\tilde{m}$.

Let P be such a partition of AD_n. Since P is simple, then for each altered row r, P divides the unremoved stones in r into two end sections, separated by the block of removed stones. (With each end section possibly empty). Increasing from n to $n+1$ adds one stone of each color to each row. To get $\phi(P)$ we simply remove from each altered row one more stone of each color, such that the configuration of each end section is preserved. Since we removed k additional stones of each color in total, $\phi(P)$ has weight transforming $\tilde{m}$, so the map is well-defined. (An example of ϕ is shown after the end of the proof).

ϕ is bijective, since the inverse map is obvious add one stone of each color such that the end section configurations are preserved) and well-defined. Well-definedness follows from the condition that each altered row has strictly greater than 1 stone removed, so the addition of stones still leaves the row an altered row.

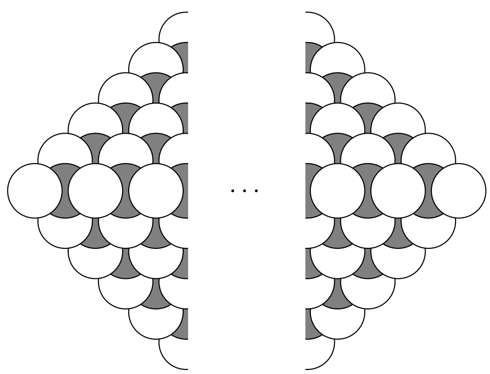
□

Example 4.8. $\phi(P)$ where P is a simple partition of AD_3.



P: $\phi(P)$:

Definition 4.9. Let AD_∞ be the following Aztec diamond pyramid extending infinitely vertically and toward the middle:



Definitions 4.10.

A **partition** of AD_∞ is a stable configuration achieved by removing stones from AD_∞, such that for each row in its decomposition, **either no stones are removed, or an infinite number of stones are removed such that only a finite number of stones remains**.

A **simple partition** of AD_∞ is a partition of AD_∞ such that the restriction of the partition to each row is simple.

For any row r of AD_n or AD_∞, define its **height** $h(r)$ as its distance from the top layer, such that the height of the top row is 0. Note that when n is finite, $h(r) = n-$ (# of white stones in r).

For any partition P of AD_n or AD_∞, define its **height**

$$h(P) = \sum_{\text{altered rows } r \text{ of } P} h(r)$$

For any partition P of AD_n or AD_∞, let

$$x(P) = \sum_{\text{altered rows } r \text{ of } P} (\#\text{ non-removed white stones in } r)$$

Equivalently,

$$x(P) = \sum_{\text{altered rows } r \text{ of } P} (\#\text{ non-removed black stones in } r)$$

Definition 4.11. Define a partition function

$$T = \sum_{P \text{ a simple partition of } AD_\infty} y_0^{x(P)+h(P)+\#\text{ altered rows}} y_1^{x(P)+h(P)}$$

Proposition 4.12. *For the conifold*

$$\lim_{n\to\infty} \tilde{F}_n = T$$

Proof. We show that if P is a simple partition of AD_n for finite n, and $m \neq 1$ is its weight, then m transforms to $\tilde{m} = y_0^{x(P)+h(P)+\#\text{ altered rows}} y_1^{x(P)+h(P)}$.

Let $m = y_0^a y_1^{a-k}$. Then m transforms to $\tilde{m} = y_0^{nk-a+k} y_1^{nk-a}$. Note that $k = \#$ altered rows. Also observe that $nk = h(P) + \sum_{\text{altered rows } r \text{ of } P}$ length of r . (Where length of $r = \#$ white stones in r before any stones are removed). Hence $nk - a = h(P) + x(P)$.

□

Remark 4.13. Comparing Definition 4.11 to Definition 3.14 from the previous section reveals that the two are indeed analogous. In the case of the previous section, $h(P)$ is always 0, and the # of altered rows is always 1.

Remark 4.14. Proposition 4.12 and Proposition 3.15 from the previous section appear to be non-analogous, due to an addition "+1" term in the earlier result. However, this appearance is false. The "+1" term happens to be built into the expression Proposition 4.12 (coming from the partition that removes no stones whatsoever), whereas the author felt there was not a clean way to "build in" this term in the case of Proposition 3.15 without seeming overly artificial.

However, after having seen the Aztec Diamond quiver, there is now a natural way to revise Section 3 in order to build this term in. Replace all the definitions regarding partitions of R_∞ with the definitions regarding partitions of A_∞ (i.e. treat R_∞ exactly the same way as A_∞). Then removing no stones at all from R_∞ would be considered a simple partition, it would have 0 altered rows, it would have height 0, and hence would contribute the term 1.

5. Conclusion

5.1. **Acknowledgments.** This research was carried out as part of the 2016 summer REU program at the School of Mathematics, University of Minnesota, Twin Cities, and was supported by NSF RTG grant DMS-1148634. The author would especially like to thank her mentor and her TA for this project, Gregg Musiker and Ben Strasser.

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