Integer partitions

Warm-up: Euler's pentagonal number theorem

$$\prod_{n=1}^{\infty} (1-q^n) = 1 + \sum_{n=1}^{\infty} (-1)^n \left( \frac{q^{\frac{n(3n-1)}{2}}}{n} + \frac{q^{\frac{n(3n+1)}{2}}}{n} \right)$$

$$\sum_{\lambda \text{ with distinct parts}} q^{1/2} (-1)^{\ell(\lambda)} \sum_{\mu \text{ with distinct parts, very special}} q^{1/2} (-1)^{\ell(\mu)}$$

Franklin's involution on distinct part \( \lambda \) has fixed points given by RHs

$$\mu = \begin{cases} 2n & \text{or} \\ n+2 & n+1 \\ n+1 & n \end{cases}$$

Want to do something similar for...

\text{Rogers-Ramanujan} \hspace{1cm} \text{THM (Me\'cMahon, Schur)}$

\# partitions of \( N \) into parts with consecutive parts differences \( \geq 2 \) = \# partitions of \( N \) into parts \( \equiv 1, 4 \mod S \)

\( e.g. \lambda = (11, 9, 5, 3, 1) \)

\( e.g. \mu = (7, 9, 9, 4, 1, 1, 1, 1) \)
As a generating function identity:

\[
\sum_{m=0}^{\infty} \frac{q^{m^2}}{(1-q^3) \cdots (1-q^m)} = \prod_{k=1}^{\infty} \frac{1}{(1-q^{3k})(1-q^{3k+1})} = \prod_{k=0}^{\infty} \frac{1}{(1-q^{3k})(1-q^{3k+2})}
\]

We'll write down something stronger.

In preparation, Sylvester proved

\[
\prod_{k=1}^{\infty} (1 + a q^k) = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k-1)}}{2} \cdot a \cdot \left( q \cdot \frac{(1{-a}q; q)_k}{(q; q)_k} + \frac{(-a q; q)_k}{(q; q)_k} \right)
\]

where \((A; q)_k = (1-A)(1-Aq) \cdots (1-Aq^{k-1})\)

\[
\sum_{\lambda \text{ with distinct parts}} \frac{q^{\lambda}}{a^k}
\]

On the RHS, look at the Durfee square of a partition with \(k\) distinct parts

\[
\frac{q^k}{2} \cdot a \cdot \left\{ \begin{array}{cl} \frac{q^k}{(q; q)_k} & \text{if } \diamond \text{ is present} \\ \frac{q^k}{(q; q)_{k-1}} & \text{if } \diamond \text{ is not present} \end{array} \right.
\]

Key Rogers–Ramanujan mod 5 identity:

The only change

\[
\prod_{k=1}^{\infty} (1 + a q^k) \sum_{m=0}^{\infty} \frac{q^{m^2} (q^5; q^5)^m}{(1-q^3) \cdots (1-q^m)} = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k-1)}}{2} a^k \cdot \frac{1}{(1{-a}q; q)_k} \cdot (q; q)_{k-1}
\]

Distinct \((a) \times \text{Diff}_2 \left( (-a) \right) \times \left( q \cdot \frac{(1{-a}q; q)_k}{(q; q)_k} + \frac{(-a q; q)_k}{(q; q)_k} \right)

\uparrow \quad \text{parts weighted by } a

\uparrow \quad \text{parts weighted by } -a

\text{PROBLEM: Find } \Phi \text{ an involution on } \text{Distinct} \times \text{Diff}_2 \text{ with fixed points}\{/\text{distinct parts}\}^{\frac{1}{2}}

\text{replace with partitions having nothing under the Durfee square}
INspirational Example:

\[
\sum_{k=1}^{\infty} \frac{(1-q^k)}{(1-q^k)} = 1
\]

\[\text{Dist} \times \text{All} \text{ has one inclusion } \phi \text{ whose fixed points are just } (\phi, \bar{\phi}),\]

\[
\begin{pmatrix}
\phi, \\
\bar{\phi}
\end{pmatrix}
\]

\[\text{moving largest part from } \lambda \text{ to } \mu \text{ or vice versa.}\]

If one wants to do Rogers-Ramanujan \( \pmod{24+3} \) due to Andrews, the extra \( q^{\frac{k^2}{2}}(-a) \) becomes \( q^{\frac{k^2}{2}}(-a) \) on RHS and on LHS the \( \sum_{m=0}^{\infty} \) term becomes an \( m \)-fold multizeta.