Volumes of combinatorial polytopes
(and Ehrhart quasi-polynomials)

\[ i(P, k) = \# kP \cap \mathbb{Z}^n \]

\( P \) a lattice polytope \( \Rightarrow \) \( i(P, k) \) is polynomial in \( k \)

\[ \sum_{k \geq 0} i(P, k) x^k = \frac{h_0^* + h_1^* x + \ldots + h_n^* x^n}{(1-x)^{n+1}} \]

\[ \sum_j h_j^* = \frac{\text{NV}(P)}{\text{volume of } P} \]

Q1: What is the volume of the Birkhoff polytope

\[ B_n = \text{convex hull of } n \times n \text{ permutation matrices} \]

**conj:** \[ P(M) = \text{the unimodal base polytope for a matroid } M \]

Then the \( h^* \)-vector of \( P(M) \) is unimodal (but not log-concave)

- the Ehrhart polynomial of all matroid polytopes
  has only positive coefficients

For a semisimple Lie \( g \), \( \gamma \) highest-weight maps \( V^\gamma \)

define \[ V^\mu \circ V^\nu = \sum_{\nu} C^{\mu \nu}_{\gamma} V^\nu \]

Clebsch-Gordan coefficients

**fact:** \[ C^{\mu \nu}_{\gamma} = \# \text{integer points in a polytope } P \text{ of a certain dimension } d \]

\[ \lim_{n \to \infty} \frac{C^{\nu \mu}_{\gamma}}{n^d} = \frac{\text{vol}(P)}{d!} \]
THM: In-types A, B, C, D

\[ C_{n_2,n_4} \text{ is a quasi-polynomial mod 2} \]

CONJ: Both polynomials mod 2 have nonnegative coefficients.