C.P.S. 4/3/2015

P. Pylyavsky

Totally positive matrices with 1's on diagonal and upper-triangular

\[
\begin{pmatrix}
1 & * & * \\
* & 1 & * \\
* & * & 1
\end{pmatrix} = e_{i_1}(t_1) \cdots e_{i_N}(t_N) \quad t_i > 0
\]

factor: if you choose a reduced word

\[ w_0 = s_{i_1} s_{i_2} \cdots s_{i_N}, \quad N = (N) \]

where

\[
e_{i}(t) = \begin{bmatrix}
1 \\
\vdots \\
t \\
1
\end{bmatrix}
\]

Chevalley generators

This factorization is unique once the reduced word is fixed.

Relations among \( s_i = (i, i+1) \)

\[
s_i s_j = s_j s_i \quad (i < j)
\]
\[
s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}
\]
\[
s_i^2 = 1
\]

Relations among \( e_i(t) \)

\[
e_i(a) e_j(b) = e_j(b) e_i(a)
\]
\[
e_i(a) e_j(b) e_i(c) = e_i(a) e_i(c) e_i(b)
\]
\[
\text{with } a' = \frac{bc}{a+c}
\]
\[
h' = \frac{aq}{a+c}
\]
\[
c' = \frac{ab}{a+c}
\]

Now do the same for periodic unitriangular matrices...

\[
\begin{bmatrix}
1 & a & 0 & 0 \\
0 & 1 & b & 0 \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
e_i(a) = \begin{bmatrix}
1 & a & 0 & 0 \\
0 & 1 & b & 0 \\
0 & 0 & 1 & c \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

They satisfy the analogous relations for \( T_n \)
A qualitative difference: If \( \sum_{j=1}^{\infty} t_j < \infty \), then \( e_{i_1}(t_1)e_{i_2}(t_2) \ldots \) converges.

Q: When are two parametrizations equal?

Braid limits

\[
\begin{array}{cccc}
 & s_0 & s_1 & s_2 \\
A_2 & \quad & \quad & \\
& e_1(a) & e_2(b) & e_3(c) & e_4(d) & e_5(e) & e_6(f) & e_7(g)
\end{array}
\]

\[
\begin{array}{cccccc}
\begin{array}{cccc}
2 & 1 & 2 & 0 & 1 & 2 & 0 \\
1 & 2 & 1 & 0 & 1 & 2 & 0 \\
0 & 2 & 0 & 1 & 0 & 2 & 0 \\
e_{i_1} & 1 & 2 & 0 & 2 & 0 & \\
\end{array}
\end{array}
\]

Say \( i \rightarrow j \) for two infinite reduced words if one can get from \( i \) to \( j \) by a braid limit. Define \( i \sim j \) if \( i \rightarrow j \) and \( j \rightarrow i \).

Let \( Q = \{ \prod_{j=1}^{\infty} e_{i_j}(t_j); t_j \in \mathbb{R}_{>0} \} \)

Given \( X \in Q \), consider all ways to write \( X = \prod_{j=1}^{\infty} e_{i_j}(t_j) \) \( \forall n \geq 0 \) \( i = (i_1, i_2, \ldots) \)

CONJ: \exists a unique \( i' \) in this set such that every other \( i \) in this set has \( i \rightarrow i' \)