General linear groups as reflection group "wannabes"

Vic Reiner
Univ. of Minnesota

Aisenstadt Lectures
CRM Univ. de Montreal
March 14-16, 2017
Three reflection group counting stories where the general linear group $\text{GL}_n$ wants in on the game...

**TALK 1:** Cycling subsets (yesterday)

**TALK 2:** Catalan numbers (today!)

**TALK 3:** Factorizations into reflections
Catalan numbers

- Catalan, q-Catalan numbers, and a cyclic sieving phenomenon

- Reflection group version

- $GL_n(F_q)$ again
Recall Catalan numbers

\[ C_n = \frac{1}{n+1} \binom{2n}{n} = \frac{(2n)!}{n!(n+1)!} \]

count many things, including triangulations of an \((n+2)\)-gon

**Example** \( n=4 \)

\[ C_4 = \frac{1}{5} \binom{8}{4} = \frac{8 \cdot 7 \cdot 6}{5 \cdot 4 \cdot 3 \cdot 2} = 14 \]

counts these:
\[ C_4 = 14 = 2 + 3 + 3 + 6 \]
These have an obvious cyclic group acting by rotations, and a cyclic sieving phenomenon (CSP):

- they are a finite set $X$,
- with a cyclic group $C = \{1, c, c^2, \ldots, c^{m-1}\}$ of order $m$ acting on $X$,
- and a polynomial $X(g)$, with

$$\# \{ x \in X : c^d(x) = x \} = X(g) \bigg|_{g = \left( e^{\frac{2\pi i d}{m}} \right)}$$
**Theorem (R. Stanton-White)**

MacMahon's $q$-Catalan number

$$C_n(q) := \frac{1}{[n+1]_q} \left[ \begin{array}{c} 2n \\ n \end{array} \right]_q$$

specialized to $q = \left( e^{\frac{2\pi i}{n+2}} \right)^d$

counts the triangulations having $\frac{n+2}{d}$-fold symmetry.

---

Recall $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$

and $\left[ \begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]!_q}{[k]!_q [n-k]!_q}$

where $[n]!_q := \left[ \begin{array}{c} n \\ q \end{array} \right]_q \left[ \begin{array}{c} n-1 \\ q \end{array} \right]_q \cdots \left[ \begin{array}{c} 2 \\ q \end{array} \right]_q \left[ \begin{array}{c} 1 \\ q \end{array} \right]_q$
Thus \( C_n(q) := \frac{1}{[n+1]_q} \begin{bmatrix} 2n \\ n \end{bmatrix}_q \\
= \frac{[2n]_q, [2n-1]_q \cdots [n+2]_q}{[n]_q, [n-1]_q \cdots [2]_q} \)

**Example:**
\[
C_4(q) = \frac{1}{[5]_q} \begin{bmatrix} 8 \\ 4 \end{bmatrix}_q = \frac{[8], [7]_q, [6]_q}{[4], [3]_q, [2]_q} \\
= 1 + q + q^3 + 2q^4 + q^5 + 2q^6 + q^7 + 8q + 9q^2 + 10q^3 + 12q^4
\]

**Remark:** \( C_n(q) \) is a polynomial in \( q \), with nonnegative integer coefficients, i.e., \( C_n(q) \in \mathbb{N}[q] \).
\[ 1 + \frac{a^3}{b} + \frac{a^2}{b} + \frac{a}{b} + \frac{a\pi i}{c} + \frac{a\pi i}{c} + \frac{a\pi i}{c} = \begin{cases} 8 & q = 1 \\ 9 & q = -1 \\ 10 \end{cases} \]

0, 2, 6

3-fold

2-fold

1-fold
More generally, there are Fuss-Catalan numbers
\[ C_n^{(m)} = \frac{1}{mn+1} \binom{mn+1}{n} \]
counting dissections of an \((mn+2)\)-gon into \((m+2)\)-gons, with a similar q-analogue, CSP.

**Example** \( m=2, n=2 \)
\[ C_2^{(2)} = \frac{1}{5} \binom{3}{2} = 3 \]
REMARK  Just as $\text{GL}_n(\mathbb{F}_q)$ has an interpretation for $q$-binomials

$$\left[ \begin{array}{c} N \\ k \end{array} \right]_q = \# \{ k \text{-dimensional} \}$$

$$\mathbb{F}_q \text{-linear subspaces} \subset \mathbb{F}_q^n \}$$

$$= \# \text{Gr}_k(k \mathbb{F}_q) = \# \text{GL}_n(\mathbb{F}_q)/P_k$$

finite Grassmannian

it also has one for $q$-Fuss-Catalans:

Rewrite it as

$$\frac{1}{[mn+1]_q} \left[ \begin{array}{c} (m+1)n \\ n \end{array} \right]_q = \frac{1}{[(m+1)n+1]_q} \left[ \begin{array}{c} (m+1)n+1 \\ n \end{array} \right]_q$$

$$= \frac{1}{[a+b]_q} \left[ \begin{array}{c} a+b \\ a \end{array} \right]_q$$

where $a = n$

$b = mn+1$, so $\gcd(a, b) = 1$
**PROPOSITION:** When $\gcd(a, b) = 1$ and $\mathbb{F}_q^\times$ inside $\text{GL}_{a+b}(\mathbb{F}_q) \cong \text{GL}_{a+b}(\mathbb{F}_q)$ acts on $\text{Gr}(a, \mathbb{F}_q)$, the subgroup $\mathbb{F}_q^\times \subset \mathbb{F}_{q+a+b}$ acts trivially, but $\mathbb{F}_{q+a+b}/\mathbb{F}_q^\times$ acts freely, with

$$\frac{1}{[a+b]_q} [a+b]_q = \# \mathbb{F}_{q+a+b} - \text{orbits on } \text{Gr}(a, \mathbb{F}_q) = \# \mathbb{F}_{q+a+b} \backslash \text{GL}_n(\mathbb{F}_q)/\text{P}_n$$

Don't know how to use this!
Recall a subgroup $W$ of $\text{GL}_n(\mathbb{F})$ is a reflection group if it is generated by reflections $t$, that is, elements whose fixed space $(\mathbb{F}^n)^t = \{v \in \mathbb{F}^n : t(v) = v\}$ is a hyperplane.

Among them are the finite subgroups $W$ of $\text{GL}_n(\mathbb{F})$ whose action on

$$S = \mathbb{F}[x_1, x_2, \ldots, x_n]$$

has the $W$-invariants polynomial

$$S_W = \mathbb{F}[f_1, f_2, \ldots, f_n]$$
**DEFINITION**: For a finite real reflection group $W$ in $GL_n(\mathbb{R})$, acting irreducibly on $\mathbb{R}^n$, let $S^W = \mathbb{R}[f_1, \ldots, f_n]$ with each $f_i$ homogeneous and of degrees $d_1 \leq \ldots \leq d_n = h_{\text{Coxeter number of } W}$.

- the **$W$-Fuss Catalan number** is
  $$\text{Cat}^{(m)}(W) := \prod_{i=1}^{n} \frac{mh + d_i}{d_i}$$

- the **$q$-$W$-Fuss Catalan number** is
  $$\text{Cat}^{(m)}(W, q) := \prod_{i=1}^{n} \frac{[mh + d_i]_q}{[d_i]_q}$$
EXAMPLE The symmetric group $G_n$ acts irreducibly on $\mathbb{R}^n$, \( \{x_1 + \cdots + x_n = 0 \} \subseteq \mathbb{R}^n \) and on $S = \mathbb{R}[x_1, \ldots, x_n] / (x_1 + \cdots + x_n)$ with $S^{G_n} = \mathbb{R}[e_2(x), e_3(x), \ldots, e_n(x)]$ where the $i$th elementary symmetric polynomial $e_i(x)$ has degree $i$.

Thus the degrees are $2 \leq 3 \leq \cdots \leq n = h$

and $\text{Cat}^n(G_n) = \prod_i \frac{h + d_i}{d_i} = \frac{(n+2)(n+3) \cdots (n+n)}{2 \cdot 3 \cdots (n)} = \frac{1}{n+1} \binom{2n}{n} = \text{(usual)} \text{ Catalan number}$
EXAMPLE (continued)

More generally, one can check

$$\text{Cat}^{(m)}(G_n, q) = \prod_i \frac{[m h + d_i]}{[d_i]}_q$$

$$= \frac{1}{[mn+1]}_q \left[ \begin{array}{c} (m+1)n \\ n \end{array} \right]_q$$

the $q$-Fuss-Catalan.
We conjectured a generalization of the CSP for rotating triangulations, proven by Eu and Fu 2008:

- triangulations
  \[\mapsto\] clusters in finite type
  \[\mapsto\] cluster algebras
- rotation
  \[\mapsto\] Fomin & Zelevinsky's deformed Coxeter element of order $h+2$
- Fuss-Catalan dissections
  \[\mapsto\] Fomin & Reading's generalized cluster cluster complexes.

Still pretty mysterious...
**THEOREM** (Berest-Etingof-Ginzburg, Gordon 2003)

\[ \text{Cat}^{m}(W) \text{ lies in } \text{IN}, \text{ and} \]
\[ \text{Cat}^{m}(W,q) \text{ lies in } \text{IN}[q]. \text{ In fact,} \]
\[ \text{Cat}^{m}(W,q) = \text{Hilb} \left( \left( \frac{S}{(\Theta_1, \ldots, \Theta_n)} \right)^W, q \right) \]

where \( \Theta_1, \ldots, \Theta_n \) are a

- homogeneous system of parameters of degree \( mh+1 \) in \( S \),
- have \( R\Theta_1 + \ldots + R\Theta_n \) \( W \)-stable,
- with same \( W \)-repns as \( R\Theta_1 + \ldots + R\Theta_n \).
Why should such magical parameters $\theta_1, \ldots, \theta_n$ exist??

In general, need subtle theory of rational Cherednik algebra $\mathcal{H}_c(W)$:

Verma module $M_c(\text{triv}) \rightarrow L_c(\text{triv})$

$s \not\parallel$ $s \not\parallel$ if $c = m + \frac{1}{h}$

$S$ $S/(\theta_1, \ldots, \theta_n)$
Existence of the magical \((\mathcal{O}_3, \ldots, \mathcal{O}_n)\) was known earlier for \(W = S_n\) (Haiman 1993, Dunkl 1998) but the arguments were tricky.

For \(W = W(B_n) = \text{hyperoctahedral group} = \{n \times n \text{ signed permutations}\}\) and \(W = W(D_n) \subset W(B_n)\), it's easy:

\(h = 2n\) or \(2(n-1)\) is even and one can take

\[(\mathcal{O}_3, \ldots, \mathcal{O}_n) = (\chi_1^{m+1}, \ldots, \chi_n^{m+1})\]
**Observation:** For $W = GL_n(F_q)$ acting on $S = F_q[x_1, \ldots, x_n]$, 

\[(\mathcal{O}_1, \ldots, \mathcal{O}_n) = (x_1^{q^m}, \ldots, x_n^{q^m})\]

- Form a homogeneous system of parameters in $S$, of degree $q^m$.
- With $\mathbb{F}_q \mathcal{O}_1 + \cdots + \mathbb{F}_q \mathcal{O}_n = \{ (c_1 x_1^{q^m} + \cdots + c_n x_n^{q^m}) : c \in \mathbb{F}_q^n \}$ $W$-stable
- Carrying same $W$-repn as $\mathbb{F}_q x_1 + \cdots + \mathbb{F}_q x_n$ \(\nabla\)
Recall our **THESIS**: 

$\text{GL}_n(\mathbb{F}_q)$ pretends to be a real reflection group with

- Coxeter number $h = q^n - 1$.
- Coxeter elements = Singer cycles

Why? Recall $S = \mathbb{F}_q[x_1, ..., x_n]$ has $S^{\text{GL}_n(\mathbb{F}_q)} = \mathbb{F}_q[f_1, ..., f_n]$ Dickson polynomials of degrees $\frac{n}{q-1} < \frac{n(q-2)}{q-1} < \frac{n(q-3)}{q-1} < ... < \frac{n}{q-1} = h$

(and Singer cycles are Springer regular elements in $\text{GL}_n(\mathbb{F}_q)$ of order $h = q^n - 1$)
FURTHER EXAMPLE

Real reflection groups have magical systems of parameters

\((\Theta_1, \ldots, \Theta_n)\) of degrees \(mh + 1\) relevant for Fuss-Catalan

\[
\begin{pmatrix}
  m = 1 \\
  \mapsto \\
  h + 1
\end{pmatrix}
\]

\[\text{relevant for Catalan}\]

...while \(G_{\chi_n}(F_q)\) has its magical systems of parameters

\((\Theta_1, \ldots, \Theta_n) = (\chi_1^m, \ldots, \chi_n^m)\)

of degrees

\[
q_0^m = (q_0 - 1) + 1
\]

\[
\begin{pmatrix}
  m = n \\
  \mapsto \\
  (q_0 - 1) + 1 = h + 1
\end{pmatrix}
\]
This suggests, taking $S = \mathbb{F}_q[x_1, \ldots, x_n]$, that we should consider
\[
\mathrm{Hilb}\left( \left( S/\left( \mathcal{D}_1, \ldots, \mathcal{D}_n \right) \right)^W, t \right)
\]
\[
= \mathrm{Hilb}\left( \left( S/\left( x_1^{m_1}, \ldots, x_n^{m_n} \right) \right)^{\mathrm{GL}_n(\mathbb{F}_q)}, t \right)
\]
as a reasonable $\mathrm{GL}_n(\mathbb{F}_q)$-analogue of $\mathrm{Cat}^{(m)}(\mathcal{W}_{\mathcal{G}})$.

But what does it look like?
It's not a product...

**CONJECTURE (Lewis-R.-Stanton 2014)**

\[
\text{Hilb} \left( \frac{S}{(x_1^m, \ldots, x_n^m)^{\text{GL}_n(F_q^m)}}, t \right)
\]

\[
= \min(n,m) \sum_{k=0}^{t} \binom{n-k}{m-k} \binom{m}{k}_{q,t}
\]

where recall

\[
\binom{m}{k}_{q,t} = (q,t)\text{-binomial} = \frac{m!_{q,t}}{k!_{q,t} (m-k)!_{q,t}}
\]

\[
= \frac{\text{Hilb}(S^{P_k}, t)}{\text{Hilb}(S^{\text{GL}_n(F_q^m)}, t)}
\]
CONJECTURE

$$\text{Hilb} \left( \left( S/\left( \chi_1^m, \ldots, \chi_n^m \right) \right)^{\text{GL}_n(\mathbb{F}_q)}, t \right)$$

$$= \sum_{k=0}^{\min(n,m)} t^{(n-k)(m-k)} \left[ \begin{array}{c} m \\ k \end{array} \right]_{q,t}$$

has only been proven for

$$\begin{cases} n = 0, 1, 2 & \text{trivial, takes real work!} \\ m = 0, 1, 2 & \text{easy, recent work of P. Goyal!} \end{cases}$$
It has a tantalizing consequence, using Gorenstein duality in
$\text{Sym}^{m}(x_1^{q^m}, \ldots, x_n^{q^m})$:

**CONJECTURE:** The divided power algebra $S^* = \text{Div}(\mathbb{F}_q^n)$ has

$$\text{Hilb}\left(\left(S^* \right)^{GL_n(\mathbb{F}_q)}, t\right) = \sum_{k \geq 0} \frac{t^n(q^k - 1)}{k! \cdot q^{k!}}$$

$$= 1 + \frac{t^n(q - 1)}{1 - t^{q - 1}} + \frac{t^n(q^2 - 1)}{(1 - t^{q - 1})(1 - t^{q^2 - 1})} + \frac{t^n(q^3 - 1)}{(1 - t^{q - 1})(1 - t^{q^2 - 1})(1 - t^{q^3 - 1})} + \ldots$$

No (good) idea yet how to prove it?
REMARK: The conjecture

\[
\text{Hilb} \left( \frac{S}{(x_1^m, \ldots, x_n^m)^{\text{GL}_n(\mathbb{F}_q)}}, t \right)
\]

\[= \sum_{k=0}^{\min(n,m)} \binom{n-k}{k} (q^m-q^k)^{\left[ \begin{array}{c} m \\ k \end{array} \right]_{q,t}}
\]

would also be consistent with a (proven) CSP involving

\[X = \text{GL}_n(\mathbb{F}_q)\text{-orbits in } \mathbb{F}_{q^m}^n\]

and action of

\[C = \langle c_1, c_2, \ldots, c_2^{m-2} \rangle = \mathbb{F}_{q^m}^X = \text{Singer cycles in } \text{GL}_m(\mathbb{F}_q)\]
Thanks again for your attention!