Toric partial orders

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Outline

1. Two geometric views on posets
   - Posets as chambers
   - Posets as digraphs
2. Two geometric views on toric posets
   - Toric posets as chambers
   - Toric posets as equivalence classes of digraphs
3. A disappointment: Toric antichains

A poset is a chamber in a graphic hyperplane arrangement

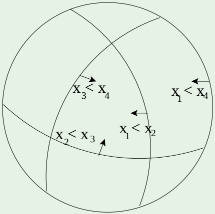
Definition

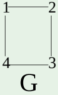
A graph G on vertex set $\{1, 2, \ldots, n\}$ has a graphic arrangement in $\mathbb{R}^n$, with hyperplanes $x_i = x_j$ for each edge $\{i, j\}$ of G.

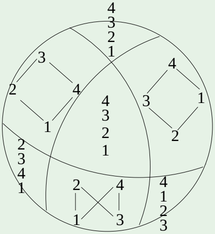
It decomposes $\mathbb{R}^n$ into connected components called chambers. They are naturally labelled by digraphs that are acyclic orientations of G, that we can think of as posets.

Example

Drawn in $\mathbb{R}^4/\mathbb{R}[1, 1, 1, 1] \cong \mathbb{R}^3$.





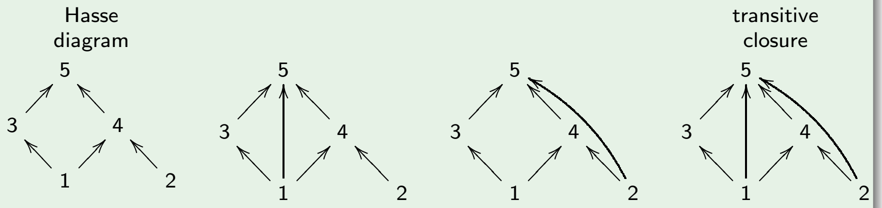


For which graphs does the poset appear as a chamber?

There's a largest such (di-)graph, its transitive closure.
There's a smallest such (di-)graph, its Hasse diagram.

The poset is represented by an acyclic orientation of any graph in between these extremes.
It appears as the same chamber in any of their graph arrangements.

Example



All of these represent the same chamber, where $x_1 < x_3, x_4 < x_5$ and $x_2 < x_4$,
but inside four different graphic arrangements within $\mathbb{R}^5$.

Transitive closure is a convex closure

Why was the smallest one, the Hasse diagram, unique?
Because transitive closure is a convex or anti-exchange closure.

Definition

An operator $2^E \longrightarrow 2^E$ on subsets of E sending $A \longmapsto \bar{A}$ is a closure if

- $A \subseteq \bar{A}$
- $\bar{\bar{A}} = \bar{A}$
- $A \subseteq B$ implies $\bar{A} \subseteq \bar{B}$

What is anti-exchange?

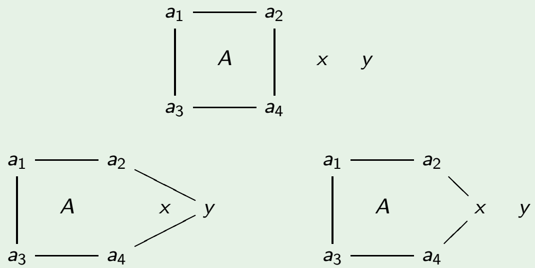
Definition

A closure is called convex closure when it satisfies this anti-exchange property:

$$x, y \notin \bar{A} \text{ and } x \in \overline{A \cup y} \text{ implies } y \notin \overline{A \cup x}$$

Example (Motivating)

Finite points sets E in Euclidean space, with $\bar{A} :=$ convexhull(A):



Example

Transitive closure on subsets of all possible digraph arrows has this anti-exchange property.

A toric poset is a chamber in a graphic hyperplane arrangement

Definition

Inside the n-torus $\mathbb{R}^n/\mathbb{Z}^n$, one has a toric graphic arrangement of the diagonal hypersurfaces/hyperplanes $x_i = x_j$ for edges $\{i, j\}$, just as before.

They decompose $\mathbb{R}^n/\mathbb{Z}^n$ into connected components, that we call toric chambers.

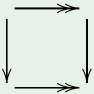
Definition

A toric poset P is any such chamber appearing in a toric graphic arrangement.

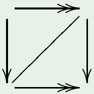
Example

For $n = 2$, it's boring: either
G is two isolated vertices, so the whole 2-torus is one chamber, or
G is the edge $\{1, 2\}$, still leaving only one chamber when you remove the diagonal $x_1 = x_2$:

$G =$ two isolated vertices



or G=an edge

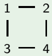


What combinatorial object parametrizes toric chambers?

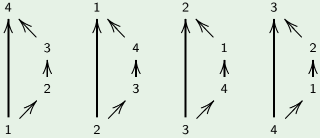
Theorem

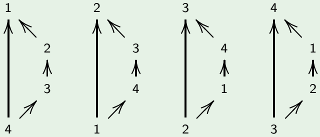
Toric chambers for G in $\mathbb{R}^n/\mathbb{Z}^n$ naturally biject with equivalence classes of acyclic orientations of G under source-to-sink moves, as studied by Mosesjan (1972), Pretzel (1986).

Example

$G =$ 

turns out to have 3 such source-to-sink equivalence classes of acyclic orientations:

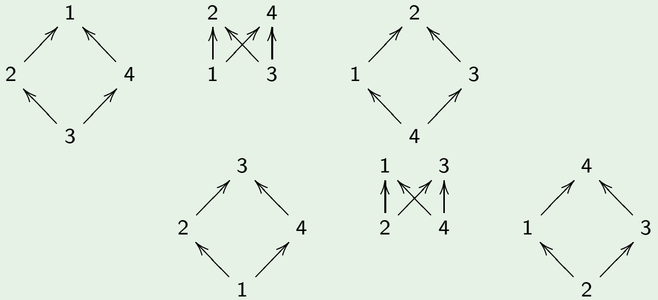
P_1 :


P_2 :


Toric chambers/posets for the 4-cycle continued...

Example

P_3 :



Aside: Tutte evaluations

Proposition

The Tutte polynomial $T_G(x, y)$ of G has these fairly well-known evaluations:

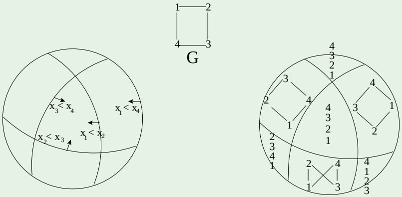
- $T_G(2, 0)$ *counts*
  acyclic orientations=chambers in the graphic arrangement=posets on G,
- $T_G(1, 0)$ *counts*
  source-to-sink classes of acyclic orientations=toric chambers=toric posets on G.

Example

The 4-cycle G has Tutte polynomial

$$T_G(x, y) = x^3 + x^2 + x + y$$

hence $T_G(2, 0) = 2^3 + 2^2 + 2 + 0 = 14$ acyclic orientations (7 each on front/back here)



falling into $T_G(1, 0) = 1^3 + 1^2 + 1 + 0 = 3$ different source-to-sink classes depicted earlier.

Some examples for $n = 3$ (Thanks, Matt!)

This graph $1 — 2 — 3$

has all 4 acyclic orientations source-to-sink equivalent.

The toric graphic arrangement consists of two hyperplanes: $x_1 = x_2$ and $x_2 = x_3$ in $\mathbb{R}^3/\mathbb{Z}^3$.

$1 \longrightarrow 2 \longrightarrow 3$

$1 \longleftarrow 2 \longleftarrow 3$

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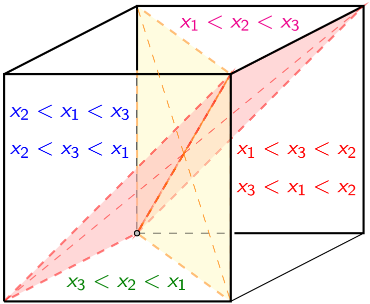
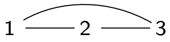


The 3-torus $\mathbb{R}^3/\mathbb{Z}^3$ is just the unit cube with opposite faces identified.
Note that the complement has only a **single toric chamber**.

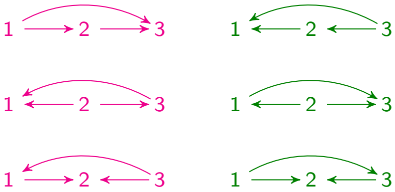
Some examples for n=3 continued...

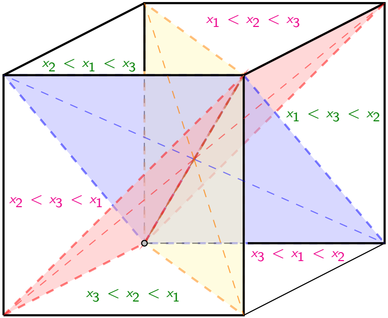
But this graph with one more edge



has 6 acyclic orientations, falling into **2 source-to-sink equivalence classes**.

The toric arrangement consists of three hyperplanes: $x_1 = x_2$, $x_2 = x_3$, and $x_1 = x_3$ in $\mathbb{R}^3/\mathbb{Z}^3$.





The 3-torus $\mathbb{R}^3/\mathbb{Z}^3$ is still just the unit cube with opposite faces identified. Note that the complement now consists of **2 toric chambers**.

A motivational digression: Coxeter groups and Coxeter elements

For a Coxeter system (W, S) with Coxeter generators $S = \{s_1, s_2, \ldots, s_n\}$, any product $c = s_{\sigma_1} s_{\sigma_2} \cdots s_{\sigma_n}$ for a permutation or total orderin σ of S is called a Coxeter element.

Proposition

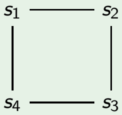
When our graph G is the (unlabelled) Coxeter diagram for (W, S), so that edges $\{s_i, s_j\}$ of G are non-commuting pairs of generators, then

- *acyclic orientations of G biject with the different possible Coxeter elements (Tits?), and*
- *the source-to-sink equivalence classes of acyclic orientations biject with the different possible W-conjugacy classes of Coxeter elements (Eriksson and Eriksson, 2010).*

The 4-cycle as a Coxeter graph

Example

Any Coxeter system whose unlabelled diagram looks like



will have 14 Coxeter elements, lying in 3 conjugacy classes, represented by

$s_1 s_2 s_3 s_4$ (which is conjugate to $s_1 \cdot s_1 s_2 s_3 s_4 \cdot s_1 = s_2 s_3 s_4 s_1$)
$s_4 s_3 s_2 s_1$
$s_1 s_2 s_4 s_3$ $(= s_1 s_4 s_2 s_3)$

Question

Can toric posets help us understand more about W-conjugacy classes?

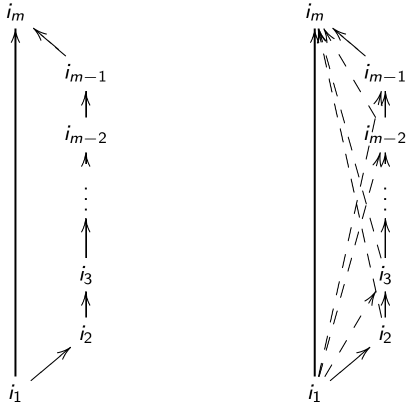
In particular, what about for the CFC (cyclically fully commutative) elements, introduced and studied by Green, Macauley, et al? These are the elements for which every cyclic shift of any of their reduced words has only commuting braid moves applicable, and they will have an associated toric heap.

Toric directed paths and toric transitive closure

Given a toric poset P, how can one tell which graphs G will have P as a chamber?

Fix any graph G having P as a toric chamber,
and any acyclic orientation whose source-to-sink class represents it.

Each time one sees a toric directed path with $m \geq 3$ as on the left below, add in the directed dotted edges shown on the right. This gives a new graph $\hat{G}$ that also has P as a chamber, along with the appropriate (source-to-sink equivalence class of) acyclic orientation of $\hat{G}$.



Toric transitive closure is a convex closure

Proposition

This graph $\bar{G}$ obtained by toric transitive closure will be independent of the choices.

The idea is that the subsets of vertices $\{i_1, i_2, \ldots, i_m\}$ that will lie on a toric directed path (call such sets the toric chains of P) doesn't change when one does a source-to-sink move. Nor does the bottom-to-top order of the elements of a toric chain, up to cyclic shift.

Similarly, one could remove all the dotted edges in any such toric directed paths, giving a smaller graph $G_{toricHasse}$ that still has P as one of it toric chambers.

Proposition

This toric Hasse diagram $G_{toricHasse}$ will be independent of the choices.

This in part comes from the following non-obvious result.

Theorem

Toric transitive closure is again a convex closure.

Extensions and total cyclic orders

Recall that the complete graph K_n on $\{1, 2, \ldots, n\}$ will have as its graphic arrangement the braid arrangement, with (Weyl) chambers $x_{\sigma_1} < \cdots < x_{\sigma_n}$ in bijection with permutations σ or total orderings of $\{1, 2, \ldots, n\}$.

Definition

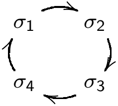
Poset P' is an extension of poset P if the chamber for P' is a subset of the chamber for P.

Proposition (Stanley 1970)

The closure of the chamber for P in the graphic arrangement is the union of the closures of the Weyl chambers for all of its extensions to total orders σ

Extensions and total cyclic orders

The toric arrangement for the complete graph K_n similarly has toric chambers in bijection with total cyclic orderings $[\sigma]$



Definition

Toric poset P' is an extension of P if the toric chamber for P' is a subset of the one for P.

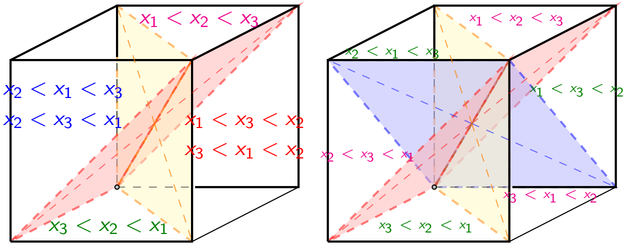
Proposition

The closure of the toric chamber corresponding to a toric poset P is the union of those corresponding to its total cyclic extensions $[\sigma]$.

Example of total cyclic extensions

Example

The toric poset for the only toric chamber of $G =$ 1 — 2 — 3 has two total cyclic extensions, the two toric chambers for $K_3 =$ 1 — 2 — 3 — 1, namely 1 → 3 → 2 → 1, 1 → 2 → 3 → 1.



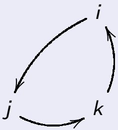
So why call them toric partial orders, and not "partial cyclic orders"?

Because that terminology is already taken, by these (related) objects...

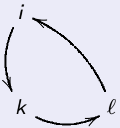
Definition

A partial cyclic order on V is a ternary relation $T \subseteq V \times V \times V$ that is...

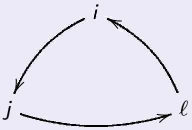
- *antisymmetric*: If $(i, j, k) \in T$ then $(k, j, i) \notin T$;
- *cyclic*: If $(i, j, k) \in T$, then $(j, k, i) \in T$;
- *transitive*: If $(i, j, k) \in T$ and $(i, k, \ell) \in T$, then $(i, j, \ell) \in T$, i.e.



together with



implies



Partial and total cyclic orders

Definition

A partial cyclic order T is total if for every $\{i, j, k\}$, either (i, j, k) or (k, j, i) appears in T.

This is easily seen to be equivalent to our total cyclic orders defined earlier.

Theorem (Megiddo, 1976)

Not every every partial cyclic order is contained in a total cyclic order (!)

Toric partial orders P do give rise to partial cyclic orders as defined above, by taking their 3-element toric chains. But one loses some info about P in the process: two different toric posets can give rise to the same partial cyclic order.

Toric antichains?

Toric chains are well-behaved, so what about antichains? There are two competing notions.

Definition

For a toric poset P on $\{1, 2, \ldots, n\}$, say that $A = \{i_1, \ldots, i_m\} \subseteq V$ is a

- combinatorial toric antichain if no distinct $i, j \in A$ lie on a common toric chain of P.
- geometric toric antichain if the subspace $\{x \in \mathbb{R}^n/\mathbb{Z}^n : x_{i_1} = \cdots = x_{i_m}\}$ intersects the (open) chamber for P.

Proposition

Geometric toric antichains are always combinatorial toric antichains, but not vice-versa.

Example (Exercise)

For a combinatorial geometric antichain that is *not* geometric, look at the 6-cycle $G = C_6$.

Dilworth's and Mirsky's theorems fail for both notions of antichain!

Recall these results for an ordinary poset P.

Theorem (Dilworth)

$$\max\{|A| : A \text{ an antichain in } P\} = \min\{\ell : P = \cup_{i=1}^{\ell} C_i, \text{ with } C_i \text{ chains in } P\}$$

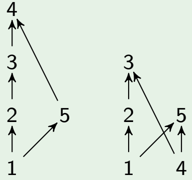
Theorem (Mirsky)

$$\max\{|C| : C \text{ a chain in } P\} = \min\{\ell : P = \cup_{i=1}^{\ell} A_i, \text{ with } A_i \text{ antichains in } P\}.$$

For toric posets, using either notion of toric antichain, one only gets the easy max$\leq$min inequality from both theorems. The inequalities can all be strict.

Example

Let G be a 5-cycle, and consider its toric poset P containing these two acyclic orientations:



The largest toric chain of P has size 2.
The largest toric antichain (of either type) has size 2.

As $|P| = 5$, it has no partition into 2 toric antichains, nor into 2 toric chains.

Thanks for listening!