Representation theory & reflection groups

Recall...

**DEFINITION:** For a group $G$, a representation $\rho$ of $G$ on a $C$-vector space $V \cong C^n$ means a group homomorphism

$$G \xrightarrow{\rho} GL(V) \cong GL_n(C)$$

**EXAMPLES IN COMBINATORICS** (they abound!)

1. **Permutation representations** = those that factor

$$G \xrightarrow{\text{inclusion}} S_n \xrightarrow{\rho_{\text{perm}}} GL_n(C)$$

Consider $\sigma \in S_5$

$$\sigma = (245)(13) \quad \Rightarrow \quad \rho_{\text{perm}}(\sigma) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix} \in GL_5(C)$$

such as $G = \langle (1,2,\ldots,n) \rangle \cong \mathbb{Z}/n\mathbb{Z}$ whose $G$-orbits were necklaces

$G = \mathbb{C}[G_2] \rightarrow \mathbb{C}[G_2]$ whose $G$-orbits were Ferrers diagrams

$G = S_n \rightarrow S_n(\mathbb{C})$ whose $G$-orbits were unlabeled graphs

or the regular representation $\rho_{\text{reg}}$:

$$G \xrightarrow{\rho_{\text{reg}}} C[G]$$ in which $\rho_{\text{reg}}(g)(h) = gh$$
(2) 1-dimensional representations \( G \longrightarrow \text{GL}_1(\mathbb{C}) = \mathbb{C}^\times \)

such as the trivial representation
\[
1 = 1_G : \quad G \longrightarrow \mathbb{C}^\times \\
g \longmapsto 1
\]

or the determinant representation
\[
\text{det} : \text{GL}(V) \longrightarrow \mathbb{C}^\times \\
g \longmapsto \text{det}(g)
\]

(3) Symmetry groups of geometric objects \( P \subset \mathbb{R}^n \)

\[
G = \text{Aut}_m(P) := \{ g \in \text{GL}_n(\mathbb{R}^n) : g(P) = P \}
\]

e.g. \( P = \text{pentagon} \)  \( G = \text{Aut}_m(P) = \langle c \rangle \cong \mathbb{Z}/4\mathbb{Z} \)

\( \text{O}_2(\mathbb{R}) \subset \text{GL}_2(\mathbb{R}) \)

orthogonal group \( \text{GL}_2(\mathbb{C}) \)
(4) **Reflection groups** =
subgroups \( G \subset O_n(\mathbb{R}) \subset GL_n(\mathbb{R}) \subset GL_n(\mathbb{C}) \)
generated by Euclidean reflections \( t \)

Good examples are
\( G = \text{Aut}_\text{lin}(P) \) for regular polytopes \( P \)
\( G \) is transitive on maximal flags of faces

- vertex \( \prec \) edge \( \prec \) polygon \( \prec \ldots \prec \) facet

\[ G = I_2(m) \rightarrow O_2(\mathbb{R}) \]
e.g. \( I_2(m) \) = symmetries of a regular \( m \)-sided polygon

\[ G = E_n \rightarrow O_{n-1}(\mathbb{R}) \]
e.g. \( E_n \) = symmetries of a regular \((n-1)\)-dimensional simplex

\( n = 4 \)

\( n = 3 \)
(4) **DEFINITION:** Say two representations \( G \rightarrow GL(V) \)
\( G \rightarrow GL(V') \)
are equivalent if there is a \( G \)-linear isomorphism \( V \simrightarrow V' \)
for which \( V \xrightarrow{\rho(g)} V \) for every \( g \in G \),
\[ \phi \downarrow \rho(g) \downarrow \]
\[ V' \xrightarrow{\rho'(g)} V' \]
i.e. \( \phi \circ \rho(g) \circ \phi^{-1} = \rho(g) \).

**QUESTION:** Can we classify in any sense, all \( G \)-representations up to equivalence?

**ANSWER:** Yes, when \( G \) is finite (and working over \( \mathbb{C} \))

In fact, the indispensable tool here are the traces that we've already been using ...

**DEF'N:** Given a representation \( G \rightarrow GL(V) = GL_n(\mathbb{C}) \)
its character \( \chi_\rho \) is the (conjugacy class function)
\[ G \rightarrow \mathbb{C} \]
\[ g \rightarrow \chi_\rho(g) := \text{Trace} (\rho(g)) \]
(meaning \( \chi_\rho(hgh^{-1}) = \chi_\rho(g) \) \( \forall h, g \in G \))
Maschke's Theorem:
One can always decompose $\rho = \bigoplus_{i=1}^{t} \rho_i$,
meaning $\rho(g) = \begin{bmatrix} \rho_1(g) & 0 & 0 & 0 \\
0 & \rho_2(g) & 0 & 0 \\
0 & 0 & \ddots & 0 \\
0 & 0 & 0 & \rho_t(g) \end{bmatrix}$,
where $V = \bigoplus_{i=1}^{t} V_i$, and where each representation $G \xrightarrow{\rho_i} \text{GL}(V_i)$ is simple/irreducible,
meaning $V_i$ has no $G$-stable subspaces except for $\{0\}$ and $V_i$ itself.

The list of (inequivalent) irreducible representations
$\{\rho_1, \rho_2, \ldots, \rho_r\}$
has size $r = \#\text{G-conjugacy classes}$.

In fact, the character $\chi_\rho$ of $\rho$ determines it up to equivalence, because the irreducible characters $\{\chi_{\rho_1}, \ldots, \chi_{\rho_r}\}$ give a $C$-basis for the vector space of all class functions $G \xrightarrow{\cdot} C$,
and this basis is orthonormal with respect to this positive definite Hermitian inner product on class functions:
$$\langle \chi_1, \chi_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \overline{\chi_2(g)}$$
(4) This means that when one decomposes
\[ \mathcal{P} = \bigoplus_{i=1}^{r} \rho_i \otimes m_i \] with irreducibles \( \rho_1, \ldots, \rho_r \).

One can compute the multiplicities \( m_i \) from
\[ \chi_{\rho} = \sum_{i=1}^{r} m_i \chi_{\rho_i} \]

\[ \Rightarrow \langle \chi_{\rho}, \chi_{\rho_i} \rangle = m_i. \]

Also, \( \langle \chi_{\rho}, \chi_{\rho} \rangle_G = \sum_{i=1}^{r} m_i^2 \),

so \( \chi_{\rho} \) irreducible \( \iff \langle \chi_{\rho}, \chi_{\rho} \rangle_G = 1 \).

\text{STANDARD EXAMPLES}

1. 1-dimensional representations \( G \to \mathbb{C}^* \)

are the same as their own character: \( \chi_{\rho} = \rho \)

Hence they are always class functions.

2. Permutation representations

\[ G \hookrightarrow S_n \xrightarrow{P_{\text{perm}}} \text{GL}(n, \mathbb{C}) \]

have \( \chi_{\rho}(\sigma) = \text{Trace}(\sigma) = \# \text{ of fixed points (1-cycles)} \)

of \( \sigma \) as a permutation.

and \( \langle \chi_{\rho}, \chi_{\sigma} \rangle = \frac{1}{|G|} \sum_{\sigma \in G} \chi_{\rho}(\sigma) \)

\[ = \frac{1}{|G|} \sum_{\sigma \in G} \# \text{ (fixed points of } \sigma) \]

\[ = \# \text{ of } G \text{- orbits on } \{1, 2, \ldots, n\} \]

\text{Burnside's lemma}
(3) The regular representation \( G \xrightarrow{\text{Reg}} G_{|G|} \rightarrow GL_{|G|}(C) \)

having \( \text{Reg}(g)(h) = gh \)

has \( \chi_{\text{reg}}(g) = \text{Trace}(\text{Reg}(g)) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{else} \end{cases} \)

and hence \( \langle \chi_{\text{reg}}, \chi_{\rho_i} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{\text{reg}}(g)} \chi_{\rho_i}(g) \)

\[ = \frac{1}{|G|} \overline{\chi_{\text{reg}}(e)} \chi_{\rho_i}(e) \]

\[ = \frac{1}{|G|} \cdot |G| \cdot \dim(V_i) = \dim(V_i) \]

**Corollary:** The regular representation \( \text{Reg} \) of \( G \)
contains every irreducible \( \rho_i \) with multiplicity \( \dim(V_i) \)

i.e. \( \text{Reg} = \bigoplus_{i=1}^{r} \rho_i \otimes \dim(V_i) \)

\[ \sum_{i=1}^{r} \dim(V_i)^2 \]

\[ |G| = \sum_{i=1}^{r} \dim(V_i)^2 \]
**Example:**

\[ G = G_3 = \{ e, (12), (13), (23), (123), (132) \} \]

\[ \begin{array}{cccc}
    & p_{\text{perm}} & & \\
\downarrow & & & \\
\text{GL}_3(\mathbb{C}) & & & \\
\end{array} \]

- \( r = 3 \) conjugacy classes

Who are the 3 irreducible representations? 

Since \( G = \langle (12), (23) \rangle \), its 1-dimensional characters \( \chi \)

are determined by the values \( \chi(s), \chi(t) \in \mathbb{C}^\times \)

and since \( s^2 = t^2 = e \), these values are in \( \{ \pm 1 \} \),

and since \( s, t \) are conjugate in \( G_3 \), they are both +1 or both -1.

This gives two 1-dimensional characters:

\[
\begin{align*}
G_3 & \rightarrow \mathbb{C}^\times \\
\text{trivial} & \quad s, t \mapsto +1 \\
\text{sgn} & \quad s, t \mapsto -1
\end{align*}
\]

Need one more irreducible \( \rho \), and

\[
|G| = \sum_{i=1}^{3} \text{dim}(\rho_i)^2
\]

\[
3! = 1^2 + 1^2 + (\text{dim}\rho)^2
\]

\[ \Rightarrow \text{dim} \rho = 2 \]
We claim the reflection representation 
\[ G_3 \xrightarrow{\text{Pref}} O_2(\mathbb{R}) \]
is irreducible, e.g., by computing its character
\[
\chi_{\text{ref}}(e) = \text{Trace } \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2
\]
\[
\chi_{\text{ref}}((ij)) = \text{Trace } \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = 0
\]
\[
\chi_{\text{ref}}((ijk)) = \text{Trace } \begin{bmatrix} 120^\circ & \text{rotation} \\ 0 & 1 \end{bmatrix} = \text{Trace } \begin{bmatrix} e^{2\pi i/3} & 0 \\ 0 & e^{-2\pi i/3} \end{bmatrix} = e + e^{-1} = -1
\]
where \( e = e^{2\pi i/3} \)

and checking 
\[
\langle \chi_{\text{ref}}, \chi_{\text{ref}} \rangle_G = \frac{1}{3!} \sum_{\sigma \in G_3} \chi_{\text{ref}}(\sigma) \chi_{\text{ref}}(\sigma)
\]
\[
= \frac{1}{6} \left( 2 \cdot 2 + 3 \cdot (1) + 2 \cdot (-1) \chi_{(1)} \right)
\]
\[
= \frac{1}{6} (4 + 2) = 1 \quad \checkmark
\]

**CONCLUSION:** The irreducible character table for \( G_3 \) is

<table>
<thead>
<tr>
<th></th>
<th>(12)</th>
<th>(13)</th>
<th>(123)</th>
<th>(132)</th>
</tr>
</thead>
<tbody>
<tr>
<td>e</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>sgn</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>Pref</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>
EXAMPLE:

The permutation representation

\[ G_3 \xrightarrow{\rho_{\text{perm}}} GL_3(\mathbb{C}) \]

must therefore be reducible.

From its character values,

\[
\begin{array}{c|ccc}
\chi_{\text{perm}} & e & (12)(13) & (132) \\
\hline
3 & 1 & 0
\end{array}
\]

one sees that \( \chi_{\text{perm}} = \chi_{11} + \chi_{\text{ref}} \)

and hence \( \rho_{\text{perm}} = 11 \oplus \rho_{\text{ref}} \).

One can see this directly from the geometry in \( \mathbb{R}^3 (\subset \mathbb{C}^3) \):

- Plane \( x_1 + x_2 + x_3 = 1 \)
- Subtract \( \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \)
- Plane \( x_1 + x_2 + x_3 = 0 \) (Carries \( \rho_{\text{ref}} \))
- Line \( \mathbb{R}[1] \) (Carries \( 11 \))