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Lecture 2

Representation theory & reflection groups

Recall...

DEFINITION: For a group G, a representation of G on a $\mathbb{C}$-vector space $V \cong \mathbb{C}^n$ means a (group) homomorphism

$$G \xrightarrow{\rho} GL(V) \cong GL_n(\mathbb{C})$$

EXAMPLES IN COMBINATORICS (they abound!)

① Permutation representations = those that factor

$$G \xhookrightarrow{\text{inclusion}} \mathfrak{S}_n \xrightarrow{\rho_{perm}} GL_n(\mathbb{C})$$

$$\sigma \longmapsto \begin{matrix} n\times n \\ \text{permutation matrix} \\ \text{of } \sigma \end{matrix}$$

e.g.

$$\begin{matrix}\sigma = (245)(13) \\ \in \mathfrak{S}_5\end{matrix} \quad \overset{\rho_{perm}}{\longmapsto} \quad \begin{array}{c c} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array} \in GL_5(\mathbb{C})$$

such as $G = \langle (1,2,\ldots,n) \rangle \hookrightarrow \mathfrak{S}_n$ whose G-orbits were necklaces
$\cong \mathbb{Z}/n\mathbb{Z}$

$G = \mathfrak{S}_k[\mathfrak{S}_\ell] \hookrightarrow \mathfrak{S}_{k\ell}$ whose G-orbits were Ferrers diagrams in  $\}k$, ℓ

$G = \mathfrak{S}_V \hookrightarrow \mathfrak{S}_{\binom{V}{2}}$ whose G-orbits were unlabeled graphs

or the regular representation ρ_{reg}:

$G \hookrightarrow \mathfrak{S}_{|G|}$ in which $\rho_{reg}(g)(h) := gh$

② 1-dimensional representations $G \longrightarrow GL_1(\mathbb{C}) = \mathbb{C}^\times$

such as the trivial representation

$$\mathbb{1} = \mathbb{1}_G : \begin{array}{ccc} G & \longrightarrow & \mathbb{C}^\times \\ g & \longmapsto & 1 \end{array}$$

or the determinant representation

$$\det : \begin{array}{ccc} GL(V) & \xrightarrow{\det} & \mathbb{C}^\times \\ g & \longmapsto & \det(g) \end{array}$$

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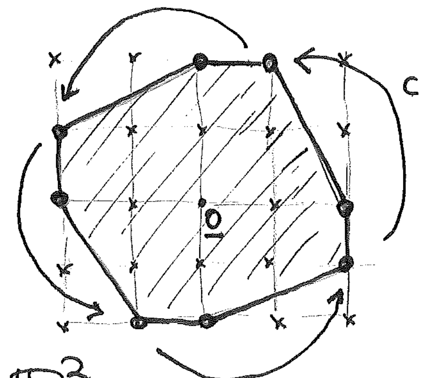
③ Symmetry groups of geometric objects $P \subset \mathbb{R}^n$

$$G = \mathrm{Aut}_{lin}(P) := \{ g \in GL_n(\mathbb{R}^n) : g(P) = P \}$$

e.g. $P =$  $\subset \mathbb{R}^2$

$G = \mathrm{Aut}_{lin}(P) = \langle c \rangle \cong \mathbb{Z}/4\mathbb{Z}$

$\subset O_2(\mathbb{R}) \subset GL_2(\mathbb{R})$

orthogonal group $\quad (\subset GL_2(\mathbb{C}))$

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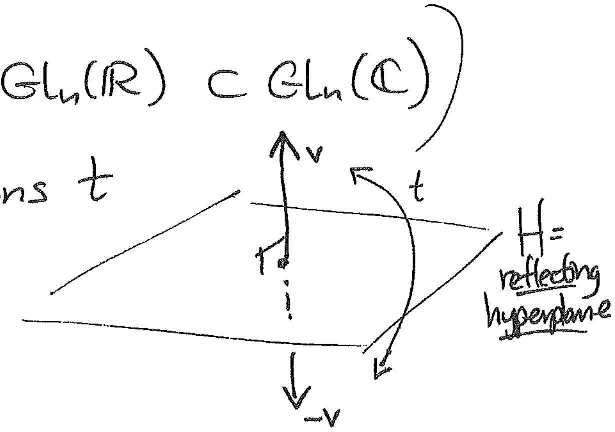
④ (real) reflection groups =

subgroups $G \subset O_n(\mathbb{R})$ $\left(\subset GL_n(\mathbb{R}) \subset GL_n(\mathbb{C})\right)$

generated by Euclidean reflections t



Good examples are

$G = \mathrm{Aut}_{lin}(P)$ for regular polytopes P

G is transitive on maximal flags of faces
vertex $\subset$ edge $\subset$ polygon $\subset \ldots \subset$ facet

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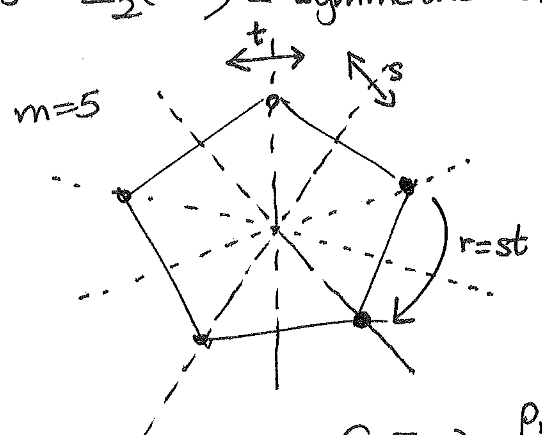
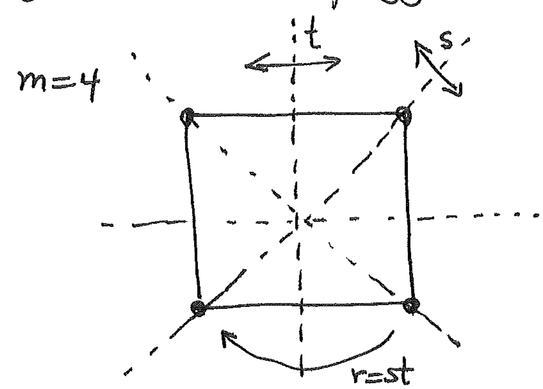
e.g. $G = I_2(m)$ = symmetries of a regular m-sided polygon





$G = I_2(m) \xhookrightarrow{\rho_{ref}} O_2(\mathbb{R})$

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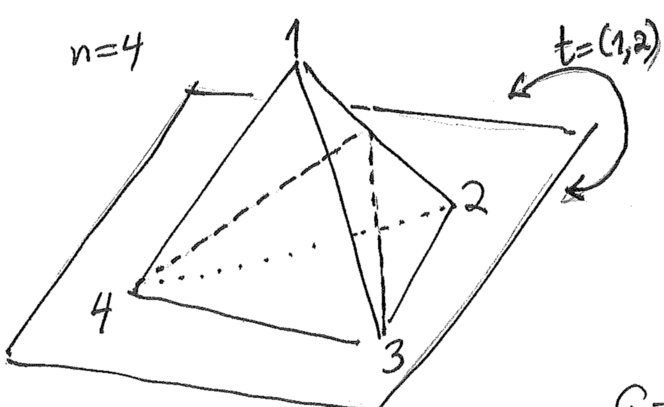
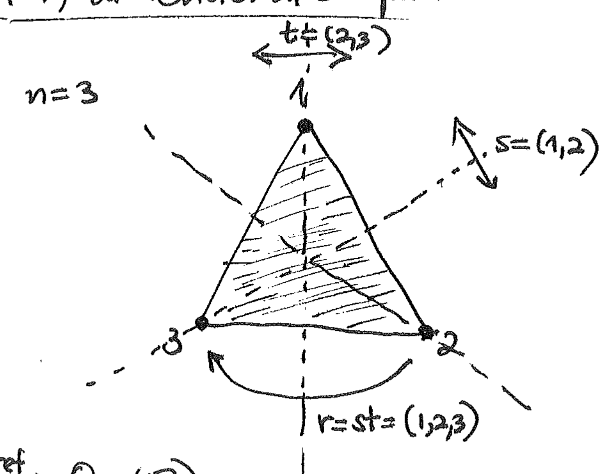
e.g. $G = \mathfrak{S}_n$ = symmetries of a regular $(n-1)$-dimensional simplex





$G = \mathfrak{S}_n \xhookrightarrow{\rho_{ref}} O_{n-1}(\mathbb{R})$

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DEFINITION: Say two representations $G \xrightarrow{\rho} GL(V)$, $G \xrightarrow{\rho'} GL(V')$ are equivalent if there is a $\mathbb{C}$-linear isomorphism $V \xrightarrow[\sim]{\varphi} V'$ for which

$$\begin{tikzcd} V \arrow[r, "\rho(g)"] \arrow[d, "\varphi", "\wr"'] & V \arrow[d, "\varphi", "\wr"'] \\ V' \arrow[r, "\rho'(g)"] & V' \end{tikzcd}$$

for every $g \in G$, i.e. $\varphi^{-1} \circ \rho'(g) \circ \varphi = \rho(g)$.

QUESTION: Can we classify, in any sense, all G-representations up to equivalence ??

ANSWER: Yes, when G is finite (and working over $\mathbb{C}$)

In fact, the indispensable tool here are the traces that we've already been using ...

DEF'N: Given a representation $G \xrightarrow{\rho} GL(V) = GL_n(\mathbb{C})$ its character χ_ρ is the (conjugacy) class function

$$\begin{aligned} G &\xrightarrow{\chi_\rho} \mathbb{C} \\ g &\longmapsto \chi_\rho(g) := \mathrm{Trace}(\rho(g)) \end{aligned}$$

meaning $\chi_\rho(hgh^{-1}) = \chi(g) \ \forall h, g \in G$

FINITE GROUP REPRESENTATION THEORY OVER $\mathbb{C}$ "REVIEW"

① Maschke's Theorem:

One can always decompose $\rho \cong \bigoplus_{i=1}^{t} \rho_i$,

meaning

$$\rho(g) = \begin{array}{c|cccc} & V_1 & V_2 & & V_t \\ \hline V_1 & \rho_1(g) & 0 & 0 & 0 \\ V_2 & 0 & \rho_2(g) & 0 & 0 \\ & 0 & 0 & \rho_3(g) & 0 \\ V_t & 0 & 0 & 0 & \rho_t(g) \end{array}$$

where $V = \bigoplus_{i=1}^{t} V_i$, and where each representation

$$G \xrightarrow{\rho_i} GL(V_i) \text{ is simple/irreducible,}$$

meaning V_i has no G-stable subspaces, except $\{\underline{0}\}$ and V_i itself.

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② The list of (inequivalent) irreducible representations

$$\{\rho_1, \rho_2, \ldots, \rho_r\}$$

has size $r = \#G$-conjugacy classes

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③ In fact, the character χ_ρ of ρ determines it up to equivalence, because the irreducible characters $\{\chi_{\rho_1}, \ldots, \chi_{\rho_r}\}$ give a $\mathbb{C}$-basis for the $\mathbb{C}$-vector space of all class functions $G \xrightarrow{f} \mathbb{C}$, and this basis is orthonormal with respect to this positive definite Hermitian inner product on class functions:

$$\langle \chi_1, \chi_2 \rangle_G := \frac{1}{|G|} \sum_{g \in G} \overline{\chi_1(g)}\, \chi_2(g)$$

④ This means that when one decomposes

$$\rho = \bigoplus_{i=1}^{r} \rho_i^{\oplus m_i} \quad \text{with irreducibles } \rho_1, \dots, \rho_r$$

one can compute the multiplicities m_i from

$$\chi_\rho = \sum_{i=1}^{r} m_i \chi_{\rho_i}$$

$$\Rightarrow \langle \chi_\rho, \chi_{\rho_i} \rangle = m_i .$$

Also, $\langle \chi_\rho, \chi_\rho \rangle_G = \sum_{i=1}^{r} m_i^2$, so χ_ρ irreducible $\Leftrightarrow \langle \chi_\rho, \chi_\rho \rangle_G = 1$

STANDARD EXAMPLES

① 1-dimensional representations $G \xrightarrow{\rho} \mathbb{C}^\times$ are the same as their own character: $\chi_\rho = \rho$

Hence they are always class functions

② Permutation representations

$$\begin{tikzcd} G \arrow[r, hook] \arrow[rr, bend right, "\rho"'] & \mathfrak{S}_n \arrow[r, "\rho_{perm}"] & GL_n(\mathbb{C}) \end{tikzcd}$$

have $\chi_\rho(\sigma) = \mathrm{Trace}(\sigma) = $ # of fixed points (1-cycles) of σ as a permutation

and $$\langle \chi_\rho, \chi_{\mathbb{1}} \rangle_G = \frac{1}{|G|} \sum_{\sigma \in G} \chi_\rho(g)$$

$$= \frac{1}{|G|} \sum_{\sigma \in G} \#(\text{fixed points of } \sigma)$$

$$\underset{\text{Burnside's lemma}}{=} \# \text{ of } G\text{-orbits on } \{1, 2, \dots, n\}$$

(3) The regular representation $G \overset{\rho_{reg}}{\hookrightarrow} \mathfrak{S}_{|G|} \longrightarrow GL_{|G|}(\mathbb{C})$

having $\rho_{reg}(g)(h) = gh$

has $\chi_{reg}(g) = \mathrm{Trace}(\rho_{reg}(g)) = \begin{cases} |G| & \text{if } g = e \\ 0 & \text{else} \end{cases}$

and hence
$$\langle \chi_{reg}, \chi_{\rho_i} \rangle_G = \frac{1}{|G|} \sum_{g \in G} \overline{\chi_{reg}(g)}\, \chi_{\rho_i}(g)$$
$$= \frac{1}{|G|} \overline{\chi_{reg}(e)}\, \chi_{\rho_i}(e)$$
$$= \frac{1}{|G|} \cdot |G| \cdot \dim_{\mathbb{C}}(V_i) = \dim_{\mathbb{C}}(V_i)$$

COROLLARY: The regular representation ρ_{reg} of G contains every irreducible ρ_i with multiplicity $\dim_{\mathbb{C}}(V_i)$

i.e. $$\rho_{reg} = \bigoplus_{i=1}^{r} \rho_i^{\oplus \dim_{\mathbb{C}}(V_i)}$$

$\Big\downarrow$ take dimensions

$$|G| = \sum_{i=1}^{r} \dim_{\mathbb{C}}(V_i)^2$$

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EXAMPLE :

$$
\begin{array}{ccccccc}
G = \mathfrak{S}_3 = \{ & e, & (12), & (13), & (23), & (123), & (132) \} \\
\rho_{perm} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
GL_3(\mathbb{C}) & \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} & \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} & \begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix} & \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix} & \begin{bmatrix}0&0&1\\1&0&0\\0&1&0\end{bmatrix} & \begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}
\end{array}
$$

$r=3$ conjugacy classes

Who are the 3 irreducible representations ?

Since $G = \langle \underset{s}{\underset{\parallel}{(12)}}, \underset{t}{\underset{\parallel}{(23)}} \rangle$, its 1-dimensional characters χ

are determined by the values $\chi(s), \chi(t) \in \mathbb{C}^\times$

and since $s^2 = t^2 = e$, these values are in $\{\pm 1\}$,

and since s,t are conjugate in $\mathfrak{S}_3$, they are both $+1$ or both -1.

This gives two 1-dimensional characters:

$$\mathfrak{S}_3 \xrightarrow{\mathbb{1} = \text{trivial}} \mathbb{C}^\times \qquad s,t \longmapsto +1$$

$$\mathfrak{S}_3 \xrightarrow{\text{sgn}} \mathbb{C}^\times \qquad s,t \longmapsto -1$$

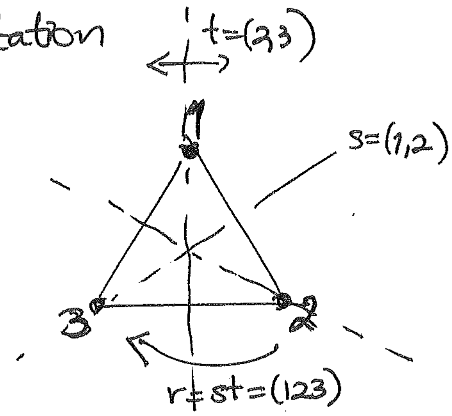
Need one more irreducible ρ, and $|G| = \sum_{i=1}^{3} \dim_{\mathbb{C}}(V_i)^2$

$$\Rightarrow 3! = 1^2 + 1^2 + (\dim_{\mathbb{C}} \rho)^2$$

$$\Rightarrow \boxed{\dim \rho = 2}$$

(9) We claim the reflection representation

$$\mathfrak{S}_3 \xrightarrow{\rho_{ref}} O_2(\mathbb{R})$$



is irreducible, e.g.

by computing its character

$$\chi_{ref}(e) = \text{Trace}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

$$\chi_{ref}((i,j)) = \text{Trace}\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} = 0$$

$$\chi_{ref}((ijk)) = \text{Trace}\begin{bmatrix} 120^\circ \\ \text{rotation} \end{bmatrix} = \text{Trace}\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix} = \zeta + \zeta^{-1} = -1$$

where $\zeta = e^{\frac{2\pi i}{3}}$

and checking $\langle \chi_{ref}, \chi_{ref} \rangle_{\mathfrak{S}} = \frac{1}{3!} \sum_{\sigma \in \mathfrak{S}_3} \overline{\chi_{ref}(\sigma)}\, \chi_{ref}(\sigma)$

$$= \frac{1}{6}\Big(\underset{e}{2 \cdot 2} + \underset{\substack{(12)\\(13)\\(23)}}{3 \cdot 0 \cdot 0} + \underset{\substack{(123)\\(132)}}{2 \cdot (-1)(-1)} \Big)$$

$$= \frac{1}{6}(4+2) = 1 \checkmark$$

CONCLUSION: The irreducible character table for $\mathfrak{S}_3$ is

	e	(12) (13) (23)	(123) (132)
$\mathbb{1}$	1	1	1
sgn	1	−1	1
ρ_{ref}	2	0	−1

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EXAMPLE:

The permutation representation

$$\mathfrak{S}_3 \xrightarrow{\rho_{perm}} GL_3(\mathbb{C})$$

must therefore be reducible.

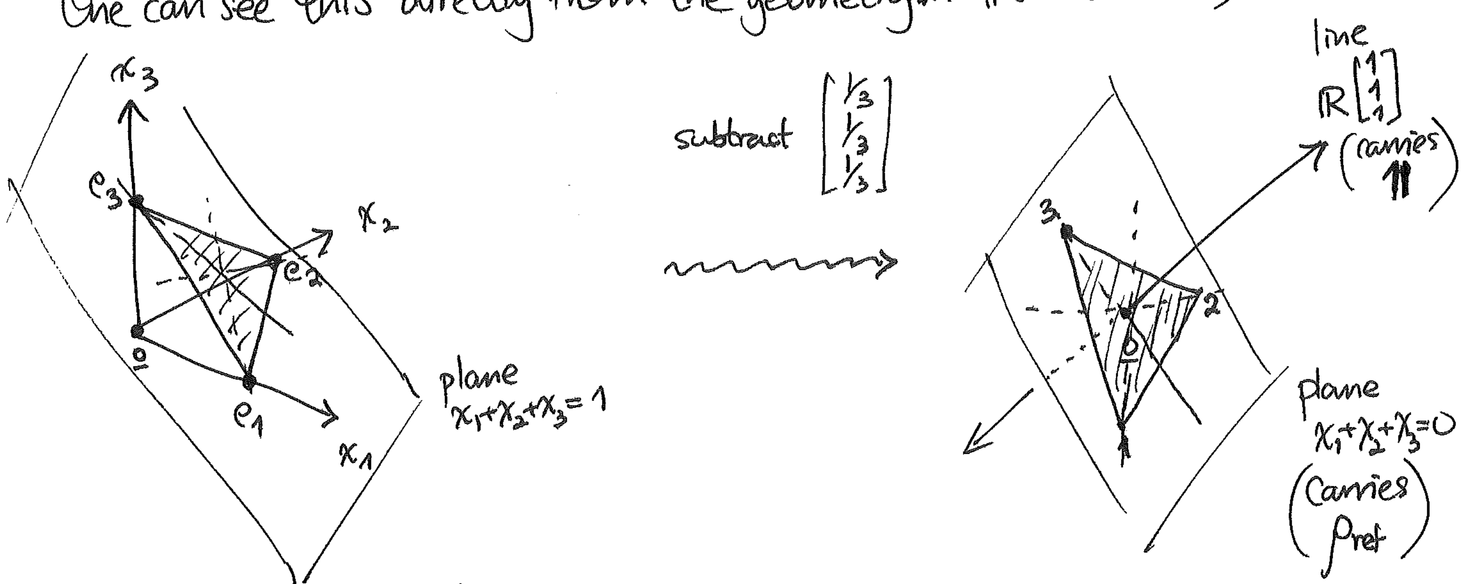
From its character values,

	e	(12) (13) (23)	(123) (132)
χ_{perm}	3	1	0

one sees that $\chi_{perm} = \chi_{\mathbb{1}} + \chi_{ref}$

and hence $\rho_{perm} = \mathbb{1} \oplus \rho_{ref}$

One can see this directly from the geometry in $\mathbb{R}^3$ $(\subset \mathbb{C}^3)$:



$\mathfrak{S}_3$ permutes coordinates here

[Generalized in EXERCISE 3]