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Lecture 3

Molien's Theorem and coinvariant algebras

Let's examine the behavior of characters of group representations under various (multi-)linear constructions ...

① DIRECT SUM

Given representations $G \xrightarrow{\rho_1} GL(V_1)$

$G \xrightarrow{\rho_2} GL(V_2)$,

we have seen $G \xrightarrow{\rho_1 \oplus \rho_2} GL(V_1 \oplus V_2)$

$$(\rho_1 \oplus \rho_2)(g)(v_1, v_2) = (\rho_1(g)(v_1), \rho_2(g)(v_2))$$

or $(\rho_1 \oplus \rho_2)(g) = \left[\begin{array}{c|c} \rho_1(g) & 0 \\ \hline 0 & \rho_2(g) \end{array}\right]$

$$\Rightarrow \chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2}$$

TENSOR PRODUCT

2. Similarly, one can create

$$G \xrightarrow{\rho_1 \otimes \rho_2} GL(V_1 \otimes V_2)$$

via $(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) = \rho_1(g)(v_1) \otimes \rho_2(g)(v_2)$

so $(\rho_1 \otimes \rho_2)(g)$ is the tensor/Kronecker product of the matrices $\rho_1(g) \otimes \rho_2(g)$

Recall for matrices $A = \begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \\ \vdots & & \ddots \end{bmatrix}$ and B

this means $A \otimes B = \left[\begin{array}{c|c|c} a_{11}B & a_{12}B & \cdots \\ \hline a_{21}B & a_{22}B & \\ \hline \vdots & & \end{array}\right]$

and $\mathrm{Trace}(A \otimes B) = \mathrm{Trace}(A)\,\mathrm{Trace}(B)$

Hence $\chi_{\rho_1 \otimes \rho_2}(g) = \chi_{\rho_1}(g)\,\chi_{\rho_2}(g)$

i.e. $\chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\,\chi_{\rho_2}$ as class functions on G

③ d^{th} TENSOR POWER

As in lecture 1, can create the d^{th} tensor power

$$T^d(V) := V^{\otimes d} = \underbrace{V \otimes V \otimes \ldots \otimes V}_{d \text{ factors}}$$

and given a G-representation $G \xrightarrow{\rho} GL(V)$

can create $G \xrightarrow{T^d(\rho)} GL(T^d(V)) = GL(V^{\otimes d})$

via $T^d(\rho)(g)(v_1 \otimes \ldots \otimes v_d) = \rho(g)(v_1) \otimes \ldots \otimes \rho(g)(v_d)$

(i.e. the diagonal action)

Thus $\chi_{T^d(\rho)}(g) = \chi_\rho(g)^d$

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④ TENSOR ALGEBRA

Putting them all together gives the tensor algebra

$$T(V) := \bigoplus_{d \geq 0} T^d(V) = \bigoplus_{d \geq 0} V^{\otimes d}$$

with a G-representation $G \xrightarrow{T(\rho)} GL(T(V))$

which now has graded character

$$\chi_{T(\rho)}(g\,;q) := \sum_{d \geq 0} q^d \cdot \chi_{T^d(\rho)}(g)$$

$$= \sum_{d \geq 0} q^d \cdot \chi_\rho(g)^d = \frac{1}{1 - q \cdot \chi_\rho(g)}$$

(5) SYMMETRIC POWERS & SYMMETRIC ALGEBRA

The d^{th} symmetric power of V is

$$\mathrm{Sym}^d(V) := V^{\otimes d} \Big/ \mathbb{C}\text{span of } \{ v_1 \otimes \cdots \otimes v_i \otimes v_{i+1} \otimes \cdots \otimes v_d - v_1 \otimes \cdots \otimes v_{i+1} \otimes v_i \otimes \cdots \otimes v_d \}$$

and denote by $v_1 \cdot v_2 \cdot \ldots \cdot v_d$ the image of $v_1 \otimes v_2 \otimes \cdots \otimes v_d$ in the quotient

so it is now commutative : $v_1 \cdot v_2 \cdot \ldots \cdot v_d = v_{\sigma(1)} \cdot v_{\sigma(2)} \cdot \ldots \cdot v_{\sigma(d)}$ $\forall \sigma \in \mathfrak{S}_d$

Because the G-action $G \xrightarrow{T^d(\rho)} GL(V^{\otimes d})$

commutes with the $\mathfrak{S}_d$-action on the positions $v_1 \otimes \cdots \otimes v_d$,

the subspace modded out above is G-stable, and

the G-action makes sense on the quotient.

That is, one obtains a G-representation

$$G \xrightarrow{\mathrm{Sym}^d(\rho)} GL(\mathrm{Sym}^d V)$$

via $\mathrm{Sym}^d(\rho)(g)\,(v_1 \cdot v_2 \cdot \ldots \cdot v_d) = \rho(g)(v_1) \cdot \rho(g)(v_2) \cdot \ldots \cdot \rho(g)(v_d)$

Putting them together, on the symmetric algebra $\mathrm{Sym}(V) := \bigoplus_{d \geq 0} \mathrm{Sym}^d(V)$

one also obtains a G-representation $G \xrightarrow{\mathrm{Sym}(\rho)} GL(\mathrm{Sym}(V))$

Q: Can we compute its graded character

$$\chi_{\mathrm{Sym}(\rho)}(g\,;q) := \sum_{d \geq 0} q^d \cdot \chi_{\mathrm{Sym}^d(\rho)}(g) \quad ?$$

PROPOSITION: For any group representation $G \xrightarrow{\rho} GL(V)$, and $g \in G$,
$$\chi_{\mathrm{Sym}(\rho)}(g\,;q) := \sum_{d \geq 0} q^d \cdot \chi_{\mathrm{Sym}^d(\rho)}(g) = \frac{1}{\det(1_V - q\cdot\rho(g))}$$

We'll prove this in EXERCISE 1, along with a famous corollary:

THEOREM (Molien 1897) Given a finite group representation $G \xrightarrow{\rho} GL(V)$ (with $V = \mathbb{C}^n$), for any other representation ψ of G, one has
$$\sum_{d \geq 0} \langle \chi_{\mathrm{Sym}^d(\rho)}, \chi_\psi \rangle_G \cdot q^d = \frac{1}{|G|} \sum_{g \in G} \frac{\overline{\chi_\psi(g)}}{\det(1_V - q\cdot\rho(g))}$$

In particular, taking $\psi = 1_G$, one obtains
$$\mathrm{Hilb}(\mathrm{Sym}(V)^G, q) := \sum_{d \geq 0} q^d \cdot \dim_{\mathbb{C}} \mathrm{Sym}^d(V)^G$$

Hilbert series for the G-fixed subalgebra $\mathrm{Sym}(V)^G$
$$= \frac{1}{|G|} \sum_{g \in G} \frac{1}{\det(1_V - q\cdot\rho(g))}$$

Note that if $V = \mathbb{C}^n$ has $\mathbb{C}$-basis $x_1, x_2, \ldots, x_n$

then $\mathrm{Sym}(V) \cong \mathbb{C}[x_1, x_2, \ldots, x_n] =: \mathbb{C}[\underline{x}]$

polynomial ring in n variables

and $\mathrm{Sym}(V)^G \cong \mathbb{C}[\underline{x}]^G$ = G-invariant subalgebra

when $\rho(G) \subset GL_n(\mathbb{C})$ acts via linear substitutions of variables

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EXAMPLE: $G = \mathfrak{S}_3 \xhookrightarrow{\rho_{perm}} GL_3(\mathbb{C}) = GL(V)$

where $V = \mathbb{C}^3$ has basis x_1, x_2, x_3

Then $\mathrm{Sym}(V) \cong \mathbb{C}[x_1, x_2, x_3]$ with $G = \mathfrak{S}_3$ permuting variables,

hence $\mathrm{Sym}(V)^G \cong \mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3}$ = symmetric polynomials

$$= \mathbb{C}[\overset{\text{degree: }1}{e_1}, \overset{2}{e_2}, \overset{3}{e_3}]$$

$$e_1 = x_1 + x_2 + x_3, \quad e_2 = x_1x_2 + x_1x_3 + x_2x_3, \quad e_3 = x_1x_2x_3$$

FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS:

$\mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n} = \mathbb{C}[e_1, e_2, \ldots, e_n]$

where e_d = d^{th} elementary symmetric function

$$= \prod_{1 \le i_1 < \cdots < i_d \le n} x_{i_1} x_{i_2} \cdots x_{i_d}$$

Hence we expect

$$\mathrm{Hilb}(\mathrm{Sym}(V)^G, q) = (1 + q + q^2 + \ldots)(1 + q^2 + (q^2)^2 + \ldots)(1 + q^3 + (q^3)^2 + \ldots)$$

$$= \frac{1}{(1-q^1)(1-q^2)(1-q^3)}$$

What does Molien's Theorem tell us?

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EXAMPLE ($G = \mathfrak{S}_3$ continued)

Recall the $\mathfrak{S}_3$-character table

	e	(12) (13) (23)	(123) (132)
$\mathbb{1}$	1	1	1
sgn	1	-1	1
χ_{ref}	2	0	-1

and hence Molien tells us that $Sym(V) = \mathbb{C}[x_1, x_2, x_3] = \mathbb{C}[\underline{x}]$ has

$$\sum_{d \geq 0} \langle \chi_{Sym^d(\rho)}, \chi_\psi \rangle_{\mathfrak{S}_3} \cdot q^d = \begin{cases} \frac{1}{3!}\left[\frac{1}{(1-q)^3} + \frac{3(1)}{(1-q^2)(1-q)} + \frac{2(1)}{(1-q^3)} \right] & \text{if } \psi = \mathbb{1} \\ \frac{1}{3!}\left[\frac{1}{(1-q)^3} + \frac{3(-1)}{(1-q^2)(1-q)} + \frac{2(1)}{(1-q^3)} \right] & \text{if } \psi = \text{sgn} \\ \frac{1}{3!}\left[\frac{2}{(1-q)^3} + \frac{3(0)}{(1-q^2)(1-q)} + \frac{2(-1)}{(1-q^3)} \right] & \text{if } \psi = \rho_{ref} \end{cases}$$

Uses part of EXERCISE 2:
For any permutation $\sigma \in \mathfrak{S}_n$,
$$\det(1_V - q \cdot \rho_{perm}(\sigma)) = \prod_{\substack{\text{cycles } C \\ \text{of } \sigma}} (1 - q^{|C|})$$

$$= \begin{cases} \frac{1}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \mathbb{1} \\ \frac{q^3}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \text{sgn} \\ \frac{q^1 + q^2}{(1-q)(1-q^2)(1-q^3)} & \text{if } \psi = \rho_{ref} \end{cases}$$

, as expected since this is $\mathrm{Hilb}(\mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3}, q) = \mathrm{Hilb}(\mathbb{C}[e_1, e_2, e_3], q)$

VERY SUGGESTIVE!

REMARK: The graded trace PROPOSITION for $G \xrightarrow{\rho} Gl(V)$

$$\chi_{\mathrm{Sym}(\rho)}(g\,;\,q) = \frac{1}{\det(1_V - q\cdot \rho(g))}$$

that implies Molien is deduced in EXERCISE 1 from a more general

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LEMMA: Given $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ any square matrix

of variables, viewed as a $\mathbb{C}(a_{ij})$-linear map $\begin{array}{ccc} V & \xrightarrow{A} & V \\ \| & & \| \\ \mathbb{C}(a_{ij})^n & & \mathbb{C}(a_{ij})^n \end{array}$,

then one has the following identity in the powerseries ring $\mathbb{C}[[a_{ij}]]$:

$$\sum_{d\geq 0} \mathrm{Trace}_{\mathrm{Sym}^d(V)}\left(\mathrm{Sym}^d(A)\right) = \frac{1}{\det(1_V - A)}$$

But this LEMMA is also equivalent to

MacMahon's MASTER THEOREM
(1916)

described and proven in EXERCISE 3,
used to prove an interesting identity in EXERCISE 4.

So what were those mysterious numerators $f^{\psi}(q)$ that appeared in $\sum_{d\geq 0} \langle \chi_{\mathbb{C}[x_1,x_2,x_3]_d}, \chi_\psi \rangle_{\mathfrak{S}_3} \cdot q^d = \frac{f^{\psi}(q)}{(1-q)(1-q^2)(1-q^3)}$

for

$\psi =$	$\mathbb{1}$	sgn	ρ_{ref}
$f^{\psi}(q) =$	1	q^3	q^1+q^2

?

They were the fake-degree polynomials that come from viewing $\mathfrak{S}_3$ as a reflection group acting on $V=\mathbb{C}^3$ and the Shephard-Todd / Chevalley theorem:

THEOREM: (Shephard-Todd 1955, Chevalley 1955) Given a finite reflection group $G \subset GL_n(\mathbb{R})$ $(\subset GL_n(\mathbb{C}) = GL(V))$ acting on $\mathrm{Sym}(V) \cong \mathbb{C}[x_1,\ldots,x_n] := \mathbb{C}[\underline{x}]$

where $x_1,\ldots,x_n$ are a basis for V,

(a) the G-invariant subalgebra $\mathbb{C}[\underline{x}]^G$ is again a polynomial algebra: $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1,\ldots,f_n]$
for some homogeneous $f_1,\ldots,f_n$, say of degrees $d_1,\ldots,d_n$,
and hence $\mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q) = \frac{1}{(1-q^{d_1})(1-q^{d_2})\cdots(1-q^{d_n})}$

(b) As G-representations,

$$\underbrace{\mathbb{C}[\underline{x}]/(f_1,\ldots,f_n)}_{=\mathbb{C}[\underline{x}]/(\underline{f}) \text{ the coinvariant algebra}} \cong \underbrace{\rho_{\text{reg}}}_{=\text{the regular representation}}$$

MORAL: For a reflection group G, the coinvariant algebra $\mathbb{C}[\underline{x}]/(\underline{f})$ gives us naturally a graded version of the regular representation!

Then using a little bit of commutative algebra
($\mathbb{C}[x_1,\ldots,x_n]$ is a Cohen-Macaulay ring;
$f_1,\ldots,f_n$ is a system of parameters, hence a regular sequence...)
one can deduce this:

COROLLARY: For a reflection group G, as above, one has an isomorphism of graded G-representations

$$\mathbb{C}[\underline{x}] \cong \underbrace{\mathbb{C}[\underline{x}]^G}_{\substack{=\mathbb{C}[f_1,\ldots,f_n] \\ \text{carrying only} \\ \text{the trivial } G\text{-rep'n } \mathbb{1} \\ \text{in all its degrees}}} \otimes \underbrace{\mathbb{C}[\underline{x}]/(\underline{f})}_{\substack{\text{the coinvariant algebra,} \\ \text{carrying a graded version} \\ \text{of regular representation}}}$$

graded tensor product, i.e.
$$(A\otimes B)_d = \bigoplus_{i+j=d} A_i \otimes B_j$$

and hence for any G-representation ψ

$$\sum_{d\geq 0} \langle \chi_{\mathbb{C}[\underline{x}]_d}, \chi_\psi \rangle_G \cdot q^d = \mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q) \cdot \sum_{d\geq 0} \langle \chi_{(\mathbb{C}[\underline{x}]/(\underline{f}))_d}, \chi_\psi \rangle_G \cdot q^d$$

$$= \frac{1}{(1-q^{d_1})\cdots(1-q^{d_n})} \cdot \underbrace{f_\psi(q)}_{\substack{=: \text{the fake-degree polynomial} \\ \text{for } \psi}}$$

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EXAMPLE: What does the coinvariant algebra for $G = \mathfrak{S}_3 \subset GL_3(\mathbb{C})$ look like?

We have seen $Sym(V) = \mathbb{C}[x_1, x_2, x_3]$

$$Sym(V)^G = \mathbb{C}[x_1, x_2, x_3]^{\mathfrak{S}_3} = \mathbb{C}[\underset{x_1+x_2+x_3}{\overset{f_1}{e_1}}, \underset{x_1x_2+x_1x_3+x_2x_3}{\overset{f_2}{e_2}}, \underset{x_1x_2x_3}{\overset{f_3}{e_3}}]$$

and hence the coinvariant algebra is

$$\mathbb{C}[\underline{x}]/(\underline{f}) = \mathbb{C}[x_1, x_2, x_3]/(e_1, e_2, e_3)$$

use $x_1+x_2+x_3=0$ to substitute $x_3 = -(x_1+x_2)$ in e_2, e_3

$$\cong \mathbb{C}[x_1, x_2]/(x_1x_2 - x_1^2 - x_1x_2 - x_2^2 - x_1x_2,\ -x_1^2x_2 - x_1x_2^2)$$

$$= \mathbb{C}[x_1, x_2]/(x_1^2 + x_1x_2 + x_2^2,\ x_1^2x_2 + x_1x_2^2)$$

and one can check that this quotient has the following $\mathbb{C}$-basis in various degrees:

degree	0	1	2	3
$\mathbb{C}$-basis:	1	x_1, x_2	x_1^2, x_1x_2	$x_1^2x_2$
$\mathfrak{S}_3$-irreducible decomposition	$\mathbb{1}$	ρ_{ref}	ρ_{ref}	sgn

$f^{\mathbb{1}}(q) = 1 = q^0$

$f^{\rho_{ref}}(q) = q^1 + q^2$

$f^{sgn}(q) = q^3$

NOTE: $\rho_{reg} = \mathbb{1} \oplus \rho_{ref} \oplus \rho_{ref} \oplus sgn$ for $G = \mathfrak{S}_3$