We want to prove this lemma from lecture...

**Lemma:** Let \( A = (a_{ij})_{i=1,...,n}^{j=1,...,n} \) be a square matrix of variables, viewed as a linear map \( \bigvee A \rightarrow V \) where \( V = \mathbb{C}(a_{ij})^n \).

Then one has an identity of power series in \( \mathbb{C}[[a_{ij}]] \)

\[
\sum_{d \geq 0} \text{Trace} \left( \text{Sym}^d V \rightarrow \text{Sym}^d V \right) = \frac{1}{\det (1 - q - A)}
\]

To prove it, start by extending the field \( \mathbb{C}(a_{ij}) \) of rational functions to any algebraically closed field \( K \supset \mathbb{C}(a_{ij}) \), and extend \( V \) to \( K^n \).

Then one can "triangulize" \( A \), that is, one can choose an ordered \( K \)-basis \( (x_1, x_2, ..., x_n) \) for \( K^n \) so that the linear map \( K^n \rightarrow K^n \) has matrix of the form

\[
\begin{bmatrix}
    x_1 & x_2 & \cdots & x_n \\
    x_1 & x_2 & \cdots & x_n \\
    \vdots & \vdots & \ddots & \vdots \\
    x_1 & x_2 & \cdots & x_n \\
\end{bmatrix}
\]

(a) Show that this \( K \)-basis \( \{x_1, x_2, ..., x_n : j_1 + j_2 + \cdots + j_n = d \} \) for \( \text{Sym}^d V \)

(b) Explain why this implies \( \text{Sym}^d A \) acting on \( \text{Sym}^d V \)

has trace \( \sum_{j_1 + j_2 + \cdots + j_n = d} a_{j_1}^{j_2} \cdots a_{j_n}^{j_n} \).

(c) Prove the lemma.

(d) Deduce the proposition that \( x_{\text{Sym}^d(p; q)} = \frac{1}{\det (1 - q - p(g))} \).

(e) Deduce Molien's theorem from the proposition.
(2a) Prove that the permutation representation \( \mathbb{C}_n \xrightarrow{\text{perm}} \mathfrak{S}_n(\mathbb{C}) = \mathbb{C}(V) \) has the property that for any \( \sigma \in \mathbb{C}_n \)
\[
\det \left( 1 - q \cdot p_{\text{perm}}(\sigma) \right) = \prod_{\text{cycles } \sigma} \left( 1 - q^{\text{cycle length}} \right)
\]

(b) Prove that for any irreducible character \( \chi_\lambda \) of \( \mathbb{S}_n \), one has
\[
\sum_{\sigma \in \mathbb{S}_n} \chi_\lambda(\sigma) \cdot q^{\text{d}} = \frac{1}{n!} \sum_{\sigma \in \mathbb{S}_n} \chi_\lambda(\sigma) \cdot P_\sigma(1, q, q^2, \ldots)
\]

where \( P_\sigma(x_1, x_2, \ldots) := \prod_{\text{cycles } \sigma} P_{\text{cycles}}(x_1, x_2, \ldots) \)

and \( P_d(x_1, x_2, \ldots) = x_1 + x_2 + \ldots \) (= the \( d \)th power sum symmetric function)

**REMARK**: For those familiar with \( \mathbb{S}_n \)-representations and the relation to symmetric functions, along with principal specializations of Schur functions \( S_\lambda(x_1, x_2, \ldots) \), as in Stanley's *Enumerative Combinatorics* Vol. 2 § 7.18, 7.21, the right side in (b) above equals
\[
S_\lambda(n, q, q^2, \ldots) = \frac{1}{(1 - q)(1 - q^2) \cdots (1 - q^n)} f^\lambda(q)
\]

where \( f^\lambda(q) := q^{\text{maj}}(\lambda) \frac{[n]!_q}{\prod_{\text{cells } x \text{ of the Ferrers diagram of } \lambda} [h(x)]_q} \)

\[
S_\lambda(n, q, q^2, \ldots) = \sum_{\text{standard Young tableaux } \xi \text{ of shape } \lambda} q^{\text{maj}(\xi)}
\]

See Stanley's *Corollaries* 7.21.3, 7.21.5 for the undefined terms here!

Thus we have two interesting expressions for the fake degree polynomials \( \Phi_\lambda(q) \) given by (**) in the case where \( G = \mathbb{S}_n \).
(3) Given a matrix \( A = (a_{ij})_{i,j=1,...,n} \in \mathbb{C}^{n \times n} \) and non-negative integers \( k = (k_1,...,k_n) \in \mathbb{N}^n \), define \( \text{per}_k(A) \) as follows: let \( \begin{bmatrix} x_1^1 & \cdots & x_1^n \\ \vdots & \ddots & \vdots \\ x_n^1 & \cdots & x_n^n \end{bmatrix} \) and \( \begin{bmatrix} y_1^1 & \cdots & y_1^n \\ \vdots & \ddots & \vdots \\ y_n^1 & \cdots & y_n^n \end{bmatrix} \) be two sets of variables related by \( \begin{bmatrix} y_1^1 \\ \vdots \\ y_n^1 \end{bmatrix} = A \begin{bmatrix} x_1^1 \\ \vdots \\ x_n^1 \end{bmatrix} \). Then \( \text{per}_k(A) \) is the coefficient of \( x_1^{k_1} \cdots x_n^{k_n} \) in \( y_1^{k_1} \cdots y_n^{k_n} \).

(a) Check that for \( A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \), \( \text{per}_{(1,1)}(A) = ad - bc \).
(b) Prove in general that \( \text{per}_{(1,1,...,1)}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)} \) (the permanent of \( A \)).

(c) Deduce from the LEMMA in Exercise (1),

\[
\text{Maclaurin's Master Theorem:} \quad \sum_{k \in \mathbb{N}^n} \text{per}_k(A) t_1^{k_1}t_2^{k_2}\cdots t_n^{k_n} = \frac{1}{\det(I_n - TA)} \quad \text{where} \quad T = \begin{bmatrix} t_1 & t_2 & 0 \\ 0 & \ddots & \vdots \\ 0 & \cdots & t_n \end{bmatrix}
\]

(4) (a) Fix \( d \in \{1,2,3,...\} \). Show that \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} d^k = 0 \) if \( n \) is odd.
(b) When \( d=1 \), show one still has \( \sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \) if \( n \) is even.
(c) When \( d=2 \), show that \( \sum_{k=0}^{n} (-1)^{k} \binom{n}{k}^2 = (-1)^m \binom{2m}{m} \) if \( n=2m \) is even.

(HINT: Interpret the left side as \( \sum \(-1)^{1A1} \), and cancel it down to \( (A,B) \): \( A,B \subset [n], 1A1+B1=n \)).
(d) When \( d=3 \), Dickson's identity says \( \sum_{k=0}^{n} (-1)^k \binom{n}{k}^3 = (-1)^m \binom{3m}{m} \) if \( n=2m \) is even.

Deduce this from Maclaurin's Master Theorem with \( A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \), by showing the left side is \( \text{per}_{(a_1,a_2,a_3)}(A) \) = coefficient of \( x_1^{2n}x_2^{2n}x_3^{2n} \) in \( (x_1^2 x_2)(x_2^2 x_3)(x_3^2 x_1) \) while the right side is the coefficient of \( t_1^{2n}t_2^{2n}t_3^{2n} \) in \( \frac{1}{\det(I_3 - TA)} = \frac{1}{1+t_1t_2+t_1t_3+t_2t_3} \).