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Lecture 4

Cyclic sieving phenomena & Springer's Theorem

Recall that in lecture 1 we proved ...

(deBruijn (1959))
THEOREM: For a permutation group $G \subseteq \mathfrak{S}_n$, consider its orbits $\mathcal{O} = \{S_1, \ldots, S_t\}$ when G acts on the Boolean algebra $2^{[n]}$, and the $\mathbb{Z}/2\mathbb{Z}$-action via complementation sending $\mathcal{O} \overset{c}{\longmapsto} c(\mathcal{O}) = \{[n] \setminus S_1, \ldots, [n] \setminus S_t\}$.

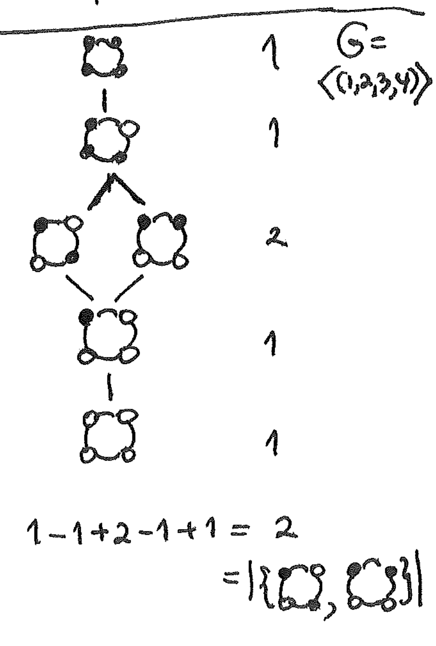
Then the poset of all G-orbits $X := 2^{[n]}/G$ and its rank-generating function
$$X(q) := r_0 + r_1 q + r_2 q^2 + \cdots + r_n q^n = \sum_{k=0}^{n} q^k \cdot \left| \binom{[n]}{k} / G \right|$$

have the property that
$$r_0 - r_1 + r_2 - \cdots \pm r_n = \#\text{self-complementary } G\text{-orbits}$$

i.e. $$[X(q)]_{q=-1} = \left| \{ x \in X : c(x) = x \} \right|$$

This is an example of what Stembridge (1994) called a "$q=-1$ phenomenon":

A set X with an action of $\mathbb{Z}/2\mathbb{Z} = \langle c \rangle$ and polynomial $X(q)$ such that
$$X(1) = |X|$$
$$X(-1) = \left| \{ x \in X : c(x) = x \} \right|$$



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More generally, one can consider sets X with actions of cyclic groups $\mathbb{Z}/m\mathbb{Z} = C = \langle c \rangle = \{e, c, c^2, \ldots, c^{m-1}\}$ for m larger than 2:

DEFINITION: (R.-Stanton-White) 2004 Say that a set X with the action of a cyclic group $C = \langle c \rangle \cong \mathbb{Z}/m\mathbb{Z}$ and a polynomial $X(q)$ exhibit a cyclic sieving phenomenon (CSP) if for every c^d in C one has

$$\left[X(q)\right]_{q=\zeta^d} = \left| \{x \in X : c^d(x) = x\} \right|$$

where $\zeta := e^{\frac{2\pi i}{m}}$

EXAMPLE (one of the first)

$X = \binom{[n]}{k}$ = k-element subsets of $[n]$

$\circlearrowleft$

$C = \mathbb{Z}/n\mathbb{Z}$
$= \langle c \rangle$
$\quad \| $
$(1, 2, \ldots, n)$

$$X(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = q\text{-binomial coefficient} = \frac{[n]!_q}{[k]!_q [n-k]!_q}$$

where $[n]!_q = [n]_q [n-1]_q \cdots [1]_q$
$[n]_q = 1 + q + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$

(= rank-generating function for $2^{[k\ell]}/G$ where $G = \mathfrak{S}_k[\mathfrak{S}_\ell]$ and $n = k+\ell$ from Lecture 1, EXERCISE 4)

THEOREM: (RSW) 2004 This $X, X(q)$ exhibits a CSP.

(EXERCISE 2 gives one of the proofs)

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e.g. $n=4$, $k=2$

$$X = \binom{[n]}{k} = \binom{[4]}{2}$$

$\circlearrowleft$

$$C = \mathbb{Z}/4\mathbb{Z} = \langle \underset{\substack{\shortparallel \\ (1,2,3,4)}}{c} \rangle = \{e, c, c^2, c^3\}$$

$$\begin{tikzcd}
\{1,4\} \arrow[r, "c"] & \{1,2\} \arrow[d, "c"] \\
\{3,4\} \arrow[u, "c"] & \{2,3\} \arrow[l, "c"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\{1,3\} \arrow[d, bend left, "c"] \\
\{2,4\} \arrow[u, bend left, "c"]
\end{tikzcd}$$

$$X(q) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}_q = \frac{[4]!_q}{[2]!_q\,[2]!_q}$$

$$= \frac{[4]_q[3]_q[2]_q[1]_q}{[2]_q[1]_q\,[2]_q[1]_q}$$

$$= \frac{[4]_q[3]_q}{[2]_q\,[1]_q}$$

$$= \frac{(1+q^1+q^2+q^3)(1+q^1+q^2)}{(1+q^1)\,(1)}$$

$$= (1+q^2)(1+q+q^2)$$

$$= 1+q+2q^2+q^3+q^4$$

$$\boxed{m=4\ ,\text{ so } \zeta = e^{\frac{2\pi i}{4}} = i}$$

$q=\zeta^0=1$: $1+1+2+1+1 = 6 = |X| \quad (=|X^e|)$

$q=\zeta^2=-1$: $1-1+2-1+1 = 2 = |\underbrace{\{\{1,3\},\{2,4\}\}}_{X^{c^2}}|$

$q=\zeta=i$ or $q=\zeta^3=i^3=-i$: $1+i-2-i+1 = 0 \quad (=|X^{c^1}|=|X^{c^3}|)$

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This first example comes from a much more general statement about reflection groups, and an enhanced version of the Shephard-Todd/Chevalley isomorphism between the coinvariant algebra and the regular representation.

(Springer 1972)

THEOREM: Given a finite reflection group $G \subset GL_n(\mathbb{C}) = GL(V)$, say that $c \in G$ is a regular element if it has an eigenvector $v \in V$ (so $c(v) = \zeta \cdot v$) lying on no reflection hyperplanes.

Then if we consider the cyclic subgroup

$$C = \langle c \rangle = \{e, c, c^2, \ldots, c^{m-1}\} \subset G,$$

one has an isomorphism of $G \times C$-representations

coinvariant algebra		regular representation
$\mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_n) \circlearrowleft$	$\cong$	$\rho_{reg} \circlearrowleft$
• G acting as before by linear substitutions		• G acting by left-translations as before: $h \xrightarrow{g} gh$
• C acting by scalar substitutions $c(x_i) = \zeta x_i \ \forall i$		• C acting by right-translations: $h \xrightarrow{c^d} hc^d$

Equivalently, for any G-representation ρ, one has

$$\chi_\rho(c) = \left[f^\rho(q) \right]_{q=\zeta}$$

the fake-degree polynomial for ρ

EXERCISE 3 asks you to check they are equivalent

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This leads to the following general CSP.

(RSW 2004)

THEOREM : When a finite reflection group $G \subset GL_n(\mathbb{C})$ acts transitively (with only one orbit) on a set X ($\cong G/H$ for some subgroup H)

and $c \in G$ is any regular element, say of order m, then one has a CSP for the action of $C = \langle c \rangle \cong \mathbb{Z}/m\mathbb{Z}$ on X with the polynomial

$$X(q) := \frac{\mathrm{Hilb}(\mathbb{C}[x_1,\ldots,x_n]^H, q)}{\mathrm{Hilb}(\mathbb{C}[x_1,\ldots,x_n]^G, q)} = \prod_{i=1}^{n}(1-q^{d_i}) \cdot \mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q)$$

since $\mathbb{C}[\underline{x}]^G = \mathbb{C}[f_1,\ldots,f_n]$ degrees $d_1,\ldots,d_n$

In other words,

$$\left[X(q)\right]_{q=\zeta^d} = \left|\{x \in X : c^d(x) = x\}\right|$$

$$= \left|\{\text{cosets } gH : c^d gH = gH\}\right|$$

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Why does this generalize our first example?

Recall there $X = \underbrace{\binom{[n]}{k}}_{\substack{k\text{-subsets} \\ \text{of } [n]}}$

$$= \overbrace{\mathfrak{S}_n}^{G} \Big/ \overbrace{\mathfrak{S}_k \times \mathfrak{S}_{n-k}}^{H}$$

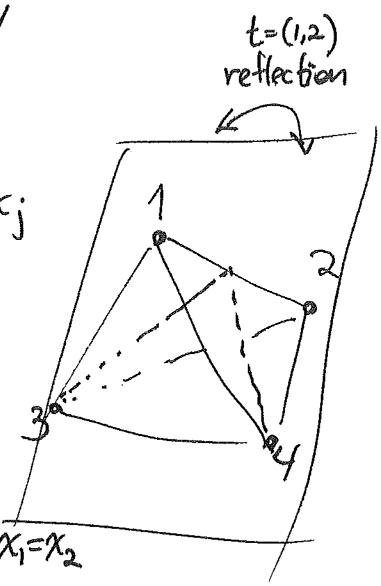
$$\mathfrak{S}_k = \mathfrak{S}_{\{1,2,\ldots,k\}} \qquad \mathfrak{S}_{n-k} = \mathfrak{S}_{\{k+1,k+2,\ldots,n\}}$$

because $G = \mathfrak{S}_n$ acts transitively on the k-subsets, and $H = \mathfrak{S}_{\{1,2,\ldots,k\}} \times \mathfrak{S}_{\{k+1,k+2,\ldots,n\}}$ is the stabilizer of a typical k-subset $\{1,2,\ldots,k\}$.

Inside $\mathfrak{S}_n$, the n-cycle $c = (1,2,\ldots,n)$ is a regular element, because when c acts on $V = \mathbb{C}^n$, it has an eigenvector

$$v = \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \end{bmatrix} \text{ where } \zeta = e^{\frac{2\pi i}{n}} : \quad c(v) = \begin{bmatrix} \zeta \\ \zeta^2 \\ \vdots \\ \zeta^{n-1} \\ 1 \end{bmatrix} = \zeta \cdot v$$

and v lies on no reflection hyperplanes $x_i = x_j$ since its coordinates are distinct.



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Hence the THEOREM implies one should have a CSP for $X = \binom{[n]}{k} = G/H$

$$\circlearrowright$$

$$C = \langle c \rangle = \{e, c, c^2, \ldots, c^{n-1}\} \cong \mathbb{Z}/n\mathbb{Z}$$
$$c = (1, 2, \ldots, n)$$

with the polynomial

$$X(q) = \frac{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q)}$$

We know $\mathbb{C}[\underline{x}]^G = \mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_n} = \mathbb{C}[e_1, e_2, \ldots, e_n]$

so $\mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q) = \dfrac{1}{(1-q)(1-q^2)\cdots(1-q^n)}$

But also $\mathbb{C}[\underline{x}]^H = \mathbb{C}[x_1, \ldots, x_n]^{\mathfrak{S}_k \times \mathfrak{S}_{n-k}} = \mathbb{C}[e_1(x_1, \ldots, x_k), e_2(x_1, \ldots, x_k), \ldots, e_k(x_1, \ldots, x_k),$
$e_1(x_{k+1}, \ldots, x_n), e_2(x_{k+1}, \ldots, x_n), \ldots, e_{n-k}(x_{k+1}, \ldots, x_n)]$

so $\mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q) = \dfrac{1}{(1-q)(1-q^2)\cdots(1-q^k) \cdot (1-q)(1-q^2)\cdots(1-q^{n-k})}$

Hence $X(q) = \dfrac{(1-q)(1-q^2)\cdots(1-q^n)}{(1-q)(1-q^2)\cdots(1-q^k) \cdot (1-q)(1-q^2)\cdots(1-q^{n-k})}$

$= \begin{bmatrix} n \\ k \end{bmatrix}_q$, as desired.

easy manipulation via $\frac{1-q^m}{1-q} = [m]_q$

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The proof idea for deducing the CSP THEOREM from Springer's THEOREM is our favorite idea of <u>comparison of traces</u>.

Start with Springer's isomorphism of $G\times C$-representations

$$\underbrace{\mathbb{C}[x_1,\ldots,x_n]/(f_1,\ldots,f_n)}^{\text{coinvariant algebra}} \cong \overbrace{\rho_{reg}}^{\text{regular representation}}$$

Take H-fixed spaces on both sides, leaving an isomorphism of C-representations

$$\left(\mathbb{C}[\underline{x}]/(\underline{f})\right)^H \cong \left(\rho_{reg}\right)^H$$

Compare the trace of c^d on the two sides:

- The left side is a graded vector space where c^d acts as the scalar $(\zeta^d)^k$ in its k^{th} graded component.
  Also, one can show the Hilbert series is $X(q) = \dfrac{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^H, q)}{\mathrm{Hilb}(\mathbb{C}[\underline{x}]^G, q)}$, so c^d acts with trace $\left[X(q)\right]_{q=\zeta^d}$.

- The right side is coset space $X = G/H$ with C-action via $c^d(gH) = c^d gH$, so c^d acts with trace $\left|\{gH : c^d gH = gH\}\right| = \left|\{x\in X : c^d(x)=x\}\right|$.