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# Hall-Littlewood polynomials $P_\lambda(x;t)$

(Refs: • Macdonald Chaps II, III
• K. Nelsen & A. Ram "Kostka-Foulkes polynomials and Macdonald spherical functions")

1. Def'n, examples, specializations
2. Hall algebra
3. Kostka-Foulkes $K_{\lambda\mu}(t)$
4. Nilpotent orbits & Lusztig's interpretation
5. Springer fibers & correspondence

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1. Recall 2 $\mathbb{Z}$-bases for $\Lambda(x_1, x_2, \ldots, x_n)$ $\left(\overset{n\to\infty}{\rightsquigarrow} \Lambda_{\mathbb{Z}}\right)$

$$m_\lambda := \sum'_{\substack{\alpha \text{ rearranging } \lambda \\ (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{N}^n}} x^\alpha = \frac{1}{m_1! m_2! \cdots} \sum_{w \in \mathfrak{S}_n} w(\underline{x}^\lambda)$$

if $\lambda = 1^{m_1} 2^{m_2} \cdots$

e.g. $m_{\square\square} = x_1^2 + x_2^2 + \cdots$

$m_{\boxminus} = x_1^1 x_2^1 + x_1^1 x_3^1 + \cdots$

Schur function

$$s_\lambda := \frac{\sum_{w \in \mathfrak{S}_n} \epsilon(w) \cdot w(\underline{x}^{\lambda+\rho})}{\prod_{1\le i<j\le n}(x_i - x_j)} = \sum_{w \in \mathfrak{S}_n} w\left(\underline{x}^\lambda \prod_{1 \le i<j\le n} \frac{x_i}{(x_i - x_j)}\right)$$

$\boxed{\rho := (n-1, n-2, \ldots, 1, 0)}$

e.g. $s_{\square\square} = x_1^2 + x_2^2 + \cdots + x_1x_2 + x_1x_3 + \cdots$

$s_{\boxminus} = x_1x_2 + x_1x_3 + \cdots$

**DEF'N:**

Hall-Littlewood polynomial

$$P_\lambda(x;t) := \frac{1}{[m_1]!_t [m_2]!_t \cdots} \sum_{w \in \mathfrak{S}_n} w\left(\underline{x}^\lambda \prod_{1\le i<j\le n} \frac{x_i - t x_j}{x_i - x_j}\right) \in \Lambda_{\mathbb{Q}(t)}(x_1, \ldots, x_n)$$

$t=1$ (obvious): $m_\lambda$

$t=0$ (obvious): $s_\lambda$

$t=-1$ (not obvious): $P_\lambda(x)$ Schur's P-functions from projective $\mathfrak{S}_n$-rep'n thy

$P_\lambda(x;q,t)$ Macdonald polynomials $\xrightarrow[q=0]{\text{(not obvious)}}$

usual

$[n]!_t := [n]_t [n-1]_t \cdots [2]_t [1]_t$

$[n]_t := 1 + t + t^2 + \cdots + t^{n-1}$

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EXAMPLES $n=2$ and $|\lambda|=2$

$$P_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(\underline{x};t) = \frac{1}{[2]!_t} \sum_{w\in\mathfrak{S}_2} w\left(x_1x_2\,\frac{x_1-tx_2}{x_1-x_2}\right) = \frac{x_1x_2}{1+t}\left[\frac{x_1-tx_2-(x_2-tx_1)}{x_1-x_2}\right]$$

$\lambda = 1^2$

$$= \frac{x_1x_2}{1+t}\left[\frac{x_1-x_2+t(x_1-x_2)}{x_1-x_2}\right] = x_1x_2 = s_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(\underline{x})$$

$$P_{\square\square}(\underline{x};t) = \frac{1}{[1]!_t} \sum_{w\in\mathfrak{S}_2} w\left(x_1^2\,\frac{x_1-tx_2}{x_1-x_2}\right) = \frac{x_1^2(x_1-tx_2)-x_2^2(x_2-tx_1)}{x_1-x_2}$$

$\lambda = 2^1$

$$= \frac{x_1^3-x_2^3-t(x_1^2x_2-x_1x_2^2)}{x_1-x_2} = x_1^2+x_1x_2+x_2^2-tx_1x_2$$

$$= s_{\square\square}(\underline{x}) - t\,s_{\begin{smallmatrix}\square\\ \square\end{smallmatrix}}(\underline{x})$$

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THM: • $P_\lambda(x_1,\dots,x_n;t)$ are stable as $n\to\infty$

• $P_\lambda(\underline{x};t) = s_\lambda(\underline{x}) + \sum_{\mu \triangleleft \lambda,\ \mu\neq\lambda} \omega_{\lambda\mu}(t)\, s_\mu(\underline{x})$ with $\omega_{\lambda\mu}(t)\in\mathbb{Z}[t]$.

dominance order

Hence $\{P_\lambda(\underline{x};t)\}$ are a $\mathbb{Z}[t]$-basis for $\Lambda_{\mathbb{Z}[t]}(\underline{x})$.

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② Hall's question:

Consider short exact sequences $0\to A\to B\to C\to 0$

where $B$ is either a finite abelian $p$-group of order $p^n$ and type $\lambda$, i.e. $B\cong \bigoplus_{i=1}^{\ell} \mathbb{Z}/p^{\lambda_i}\mathbb{Z}$, $|\lambda|=n$

or an $\mathbb{F}_q$-vector space of dim $n$ with a nilpotent endomorphism $T$ of type $\lambda$, i.e. Jordan form $\bigoplus_{i=1}^{\ell} \underbrace{J_{\lambda_i}}_{\text{nilpotent Jordan block}}$, $|\lambda|=n$

stabilizing $A$

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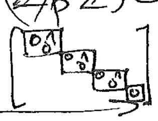
e.g. $\lambda = $ (2,2,2,1) means $B\cong (\mathbb{Z}/p^2\mathbb{Z})^3 \oplus \mathbb{Z}/p\mathbb{Z}$

or $T=$  $B \xrightarrow{T} B$

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Then $A, C$ have types $\mu, \nu$ for some $|\mu|+|\nu|=n$

Not hard to show $\mu\subseteq\lambda$, and then $\nu\subseteq\lambda$ via Pontryagin duality $B^* := \mathrm{Hom}(B,\mathbb{C}^\times) \cong B$

$0\leftarrow A^*\leftarrow B^*\leftarrow C^*\leftarrow 0$

or $\mathbb{F}_q$-vector space duality $B^* = \mathrm{Hom}_{\mathbb{F}_q}(B,\mathbb{F}_q)$

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Q: Fixing a B of type $\lambda$, how many A of type $\mu$ are there inside B, such that C is of type $\nu$?

Call this $g^{\lambda}_{\mu\nu}(q)$ or $g^{\lambda}_{\mu\nu}(p)$.

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THM (Hall) There exist polynomials $g^{\lambda}_{\mu\nu}(t) \in \mathbb{Z}[t]$ with $g^{\lambda}_{\mu\nu}(q) = g^{\lambda}_{\mu\nu}(t)\big|_{t=q}$

$$g^{\lambda}_{\mu\nu}(t) = c^{\lambda}_{\mu\nu} \cdot t^{n(\lambda)-(n(\mu)+n(\nu))} + \left(\text{lower order terms in } t\right)$$

where $n(\lambda) :=$ sum of these entries

$$\lambda: \quad \begin{array}{|c|c|c|c|} \hline 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 \\ \cline{1-3} 2 & 2 & 2 \\ \cline{1-3} 3 \\ \cline{1-1} \end{array}$$

and, in fact, $\tilde{g}^{\lambda}_{\mu\nu}(t) := t^{n(\lambda)-(n(\mu)+n(\nu))} g^{\lambda}_{\mu\nu}(t^{-1})$

are the $\{P_\lambda(x;t)\}$ structure constants:

$$\boxed{P_\mu P_\nu = \sum_\lambda \tilde{g}^{\lambda}_{\mu\nu}(t) P_\lambda}$$

$$\Big\downarrow t=0$$

$$\left( s_\mu s_\nu = \sum_\lambda c^{\lambda}_{\mu\nu} s_\lambda \right)$$

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I used SAGE to compute this

e.g. $P_{(1,1)} P_{(2,1)} = 1 \cdot P_{(3,2)} + (t+1) P_{(3,1,1)} + (t+1) P_{(2,2,1)} + (t^2+t+1) P_{(2,1,1,1)}$

$n(\mu) = 1$, $n(\nu) = 1$; $n(3,2) = 2$, $n(3,1,1) = 3$, $n(2,2,1) = 4$, $n(2,1,1,1) = 6$

$\rightsquigarrow 1$, $\quad 1+q$, $\quad q+q^2$, $\quad q^2+q^3+q^4$

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(3) Kostka-Foulkes $K_{\lambda\mu}(t)$ is defined by

$$s_\lambda(x) = \sum_\mu \underbrace{K_{\lambda\mu}(t)}_{\in \mathbb{Z}[t]} P_\mu(x;t)$$

$\in \mathbb{Z}[t]$ since $\{P_\mu\}$ are a $\mathbb{Z}[t]$-basis for $\Lambda_{\mathbb{Z}[t]}(x)$

e.g. $s_{(3,1)}(x) = 1 \cdot P_{(3,1)} + t P_{(2,2)} + (t^2+t) P_{(2,1,1)} + (t^5+t^4+t^3) P_{(1,1,1,1)}$

$\begin{smallmatrix}1&1&1\\2\end{smallmatrix}$; $\quad \begin{smallmatrix}1&1&2\\2\end{smallmatrix}$; $\quad \begin{smallmatrix}1&1&3\\2\end{smallmatrix}\ \begin{smallmatrix}1&1&2\\3\end{smallmatrix}$; $\quad \begin{smallmatrix}1&2&3\\4\end{smallmatrix}\ \begin{smallmatrix}1&2&4\\3\end{smallmatrix}\ \begin{smallmatrix}1&3&4\\2\end{smallmatrix}$

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THM: (Lascoux - Schützenberger)

$$K_{\lambda\mu}(t) = \sum_{\substack{\text{~~col-strict tableaux~~ } T \\ \text{of shape } \lambda \\ \text{content } \mu}} t^{\text{charge}(T)} \in \mathbb{N}[t], \text{ monic of degree } n(\mu)-n(\lambda)$$

(and zero unless $\lambda \trianglerighteq \mu$)

for an interesting statistic called charge(T), generalizing major index for standard tableaux $\mu = 1^n$.

## 4 Nilpotent orbits

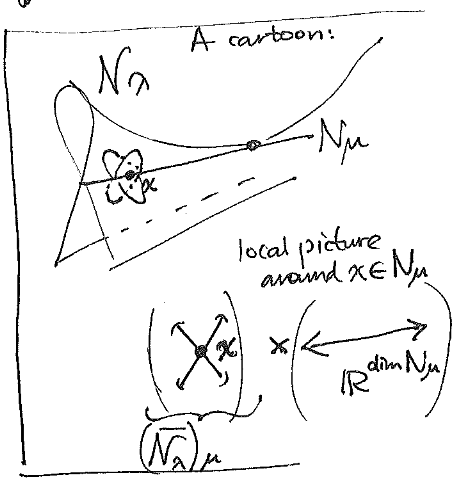
$$\underset{GL_n(\mathbb{C})}{\overset{\|}{G}} \text{ acts on } \underset{\mathfrak{gl}_n(\mathbb{C})}{\overset{\|}{\mathfrak{g}}} = \mathbb{C}^{n\times n} \text{ via conjugation } g.x := gxg^{-1}$$

$\cup$ subvariety

$$\mathcal{N} := \left\{\begin{matrix}\text{nilpotent}\\ \text{matrices}\end{matrix}\right\} \underset{\substack{\uparrow\\ G\text{-orbit}\\ \text{decomp}}}{=\!=} \bigsqcup_{\lambda \vdash n} \underbrace{\mathcal{N}_\lambda}_{\substack{\text{nilpotents}\\ \text{of Jordan}\\ \text{type } \lambda}}$$

One has $\overline{\mathcal{N}_\mu} \subseteq \overline{\mathcal{N}_\lambda} \iff \mu \underset{\substack{\uparrow\\ \text{dominance}}}{\trianglelefteq} \lambda$

and the $\{\mathcal{N}_\lambda\}$ give a stratification of $\mathcal{N}$




THM (Lusztig)

$$t^{n(\mu)-n(\lambda)} K_{\lambda\mu}(t^{-1}) = \sum_{i\geq 0} t^i \cdot \dim_{\mathbb{C}} \underbrace{IH}_{\substack{\text{intersection}\\ \text{homology}}}^{2i}\Big(\underbrace{(\overline{\mathcal{N}_\lambda})_\mu}_{\substack{\text{link of } x\in \mathcal{N}_\mu\\ \text{inside } \mathcal{N}_\lambda}}\Big)$$

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## 5 Springer fibers

THM: (Hotta-Springer) Fix $\mu$.

$$\sum_\lambda \underbrace{\tilde{K}_{\lambda\mu}(t)}_{t^{n(\mu)}K_{\lambda\mu}(t^{-1})} \cdot s_\lambda = \sum_{i\geq 0} t^i \cdot \underbrace{ch\, H}_{\substack{\text{ordinary}\\ \text{cohomology}}}^{2i}(\mathcal{B}_\mu ; \mathbb{Q})$$

$\Big\} t=1$

$$\sum_\lambda K_{\lambda\mu} s_\lambda = h_\mu = \underset{\substack{\uparrow\\ \mathfrak{S}_n\text{-}\\ \text{Frobenius}\\ \text{characteristic}}}{ch}\big(\mathbb{C}[\mathfrak{S}_n / \underset{\mathfrak{S}_{\mu_1}\times \mathfrak{S}_{\mu_2}\times\cdots}{\overset{\|}{\mathfrak{S}_\mu}}]\big)$$

where $\mathcal{B}_\mu$ = Springer fiber over any $x \in \mathcal{N}_\mu$

$= \left\{\begin{matrix}\text{complete flags}\\ 0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{C}^n\\ \text{preserved by } \exp(x) \text{ i.e. } \exp(x)\cdot V_i \subset V_i \ \forall i\end{matrix}\right\}$

In particular, the top nonvanishing cohomology $H^{2n(\mu)}(\mathcal{B}_\mu ; \mathbb{Q})$ carries the $\mathfrak{S}_n$-irreducible for $\mu$

(not true in other types – one needs the component group $Z_G(x)/Z_G(x)^\circ$ to split into W-irreducibles)

(5) One has

$$\mathcal{B}_\mu \hookrightarrow \mathcal{B}_{1^n} = \mathcal{B} = \{\text{complete flag manifold}\}$$

$$H^\bullet(\mathcal{B}_\mu;\mathbb{Q}) \cong R_\mu := \mathbb{Q}[x_1,\dots,x_n]/I_\mu$$

Tanisaki's presentation

$I_\mu$ generated by certain elementary symmetric functions $\{e_j(\underbrace{x_{i_1},\dots,x_{i_s}}_{\text{all possible subsets of size } s})\}$

$$H^\bullet(\mathcal{B};\mathbb{Q}) \twoheadrightarrow H^\bullet(\mathcal{B}_\mu;\mathbb{Q})$$

$$H^\bullet(\mathcal{B};\mathbb{Q}) \cong \mathbb{Q}[x_1,\dots,x_n]/\underbrace{(e_1(\underline{x}),e_2(\underline{x}),\dots,e_n(\underline{x}))}_{I_{1^n}:=}$$

Borel's presentation

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EXAMPLE:

$\mu = (2,1,1)$ $\quad n(\mu)=3$

| $\lambda$ | $K_{\lambda\mu}(t)$ | $\tilde{K}_{\lambda\mu}(t)=t^3K_{\lambda\mu}(t^{-1})$ |
|---|---|---|
| $(2,1,1)$ | $1$ | $t^3$ |
| $(2,2)$ | $t$ | $t^2$ |
| $(3,1)$ | $t^2+t$ | $t+t^2$ |
| $(4)$ | $t^3$ | $1$ |

$$H^\bullet(\mathcal{B};\mathbb{Q}) = \mathbb{Q}[x_1,x_2,x_3,x_4]/(e_1(\underline{x}),e_2(\underline{x}),e_3(\underline{x}),e_4(\underline{x}))$$

| deg | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $=\mathbb{Q}\{$ | $1,$ | $x_1,x_2,x_3,$ | $x_1^2, x_1x_2,\dots$ | $\dots$ | $\dots$ | $\dots$ | $\dots\}$ |
| | $s_{(4)}$ | $s_{(3,1)}$ | $s_{(3,1)}$ | $s_{(3,1)}$ | $s_{(2,1,1)}$ | $s_{(2,1,1)}$ | $s_{(1,1,1,1)}$ |
| | | | $s_{(2,2)}$ | $s_{(2,1,1)}$ | $s_{(2,2)}$ | | |

$$\downarrow$$

$$H^\bullet(\mathcal{B}_{(2,1,1)};\mathbb{Q}) = \mathbb{Q}[x_1,x_2,x_3,x_4]/(e_1(\underline{x}),e_2(\underline{x}),e_3(\underline{x}),e_4(\underline{x}), e_3(x_1,x_2,x_3), e_3(x_1,x_2,x_4), e_3(x_1,x_3,x_4), e_3(x_2,x_3,x_4))$$

| deg | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $=\mathbb{Q}\{$ | $1,$ | $x_1,x_2,x_3$ | $x_1^2, x_1x_2$ | $\dots\}$ |
| | $s_{(4)}$ | $s_{(3,1)}$ | $s_{(3,1)}$ | $s_{(2,1,1)}$ |
| | | | $s_{(2,2)}$ | |

one copy of $s_\mu$ in top degree $t^{n(\mu)}$

$\mu = (2,1,1)$

$\mu^t = (3,1)$

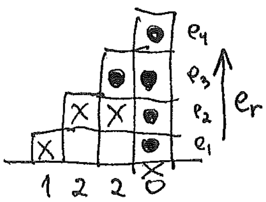
$=(0,0,1,3)$

partial sums

$(0,0,1,4)$

Subtract
$$\begin{array}{r} (1,2,3,4) \\ -(0,0,1,4) \\ \hline (1,2,2,0) \end{array} \rightsquigarrow$$

Tanisaki's ideal $I_\mu$ is gen'd by $e_r(\{x_i : i\in S\})$ for certain $r$ and cardinalities $|S|$




1 2 3 4

size of variable set $S$ in $e_r(\{x_i : i\in S\})$