Let's classify finite subgroups $G \subset \text{SL}_2(\mathbb{C}) = \text{SL}(V)$, $V = \mathbb{C}^2$ by proving two things:

**THM:** If a finite graph has a vertex-labeling $f : V \to \{1, 2, 3, \ldots \}$ which is **additive**, $2f(v) = \sum_{w \sim v} f(w)$, then $G$ is an ADE (simply-laced affine) Dynkin diagram labeled as follows, up to scaling:

$$
\begin{array}{c}
A_n, \text{ except } n = 2 \\

n = 8, 9, 10, 11, 12 \\

n = 7, 6, 5, 4, 3, 2 \\

E_8, E_7, E_6, D_n, n \geq 4 \\

\end{array}
$$

**McKay's ADE Conjecture (1982):** For a finite subgroup $G \subset \text{SL}_2(\mathbb{C})$, its McKay quiver $\Gamma$ having vertex set $\text{Inv}(G) = \{X_0, X_1, \ldots, X_N \}$, and $m_{ij}$-arcs $X_i \leftrightarrow X_j$ (so it corresponds to a graph $\Gamma$) is always loopless, and $m_{ij} = m_{ji} = \langle i, e \rangle$ (so it corresponds to a graph $\Gamma$) is additive, so $G$ is ADE as above.

**Example:** $G = \mathbb{Z} \times \mathbb{Z} \hookrightarrow \text{SL}_2(\mathbb{C}) = \text{SL}(V)$ has $\text{Inv}(G) = \{X_0, X_1, X_2, \ldots, X_{n-1} \}$, where $X_i(g) = e^{2\pi i \langle i, g \rangle}$ and the McKay quiver for $V$ is $X_{n-1} \leftrightarrow X_n \leftrightarrow X_0 \leftrightarrow X_1 \leftrightarrow X_2$.
proof of Thm 1:

- If $\Gamma$ has a cycle, it is that cycle:

\[
2f(v_i) = v_{i-1} + v_{i+1} + \ldots \mod k \geq v_{i-1} + v_{i+1}
\]

\[
\Rightarrow 2\sum_{i=0}^{k-1} f(v_i) \geq \sum_{i=0}^{k-1} f(v_i) + \sum_{i=0}^{k-1} f(v_i)
\]

\[
\Rightarrow \text{equality everywhere, so no neighbors outside the cycle,}
\]

So WLOG, $\Gamma$ is a tree.

- There are no vertices of degree $\geq 5$, and if one has degree 4, then $\Gamma = D_4$.

\[
\Rightarrow 2f(v) = f(v) + \ldots + f(v) + \ldots \geq 2f(v)
\]

\[
\Rightarrow \frac{d}{4} f(v) \leq \sum_{i=1}^{d} f(v) \leq f(v)
\]

\[
\Rightarrow \text{contradiction unless } d=4 \text{, and if } d=4 \text{ it forces equality everywhere, so no other vertices.}
\]

So WLOG, $\Gamma$ has max degree $\leq 3$.

- There exist vertices of degree 3, else $\Gamma$ is a path, forcing this labeling:

\[
\text{D}_4
\]

Contradiction to finiteness!

- If there are 2 vertices of degree 3, connected by a path, then there is nothing else, i.e. it is $D_4$:

\[
\text{by this calculation:}
\]
So WLOG $\Gamma$ has only one deg 3 vertex.

It looks like this:

$\begin{align*}
2a \Rightarrow x_2 = x_2 - a \Rightarrow x_3 = x_2 - x_4 \Rightarrow x_4 = x_3 + x_5 \\
2b \Rightarrow x_4 = x_5 \\
2c \Rightarrow x_5 = x_6 + x_7 \\
2d = 0
\end{align*}$

Also $\begin{align*}
2x_1 = a + b + x_2 \\
x_2 = x_2 - x_3 \\
x_3 = x_2 - x_4 \\
x_4 = x_3 + x_5 \\
x_5 = x_6 + x_7 \\
x_6 = c + d + x_{10}
\end{align*}$

$\begin{align*}
2\sum x_i = \frac{a+b+c+d+2\sum x_i}{3} = (x_1 + x_4) \\
\Rightarrow x_1 + x_4 = a + b + c + d \geq x_1 + x_4 \\
\text{forces equality, so no other vertices}
\end{align*}$

Then $2x = 3x - (a+b+c)$

i.e. $\begin{align*}
x &\leq 3c \\
\Rightarrow k \leq 3
\end{align*}$

- If $k=3$, then $x=3c=ab+c$ forces $a=b=c$ and $\Gamma = E_6$

- If $k=2$, then $\begin{align*}
\Gamma &= a-2a-3a--(i-a) \\
b &= 2a--(j-b) \\
c &= 2a+2b=4b
\end{align*}$

and $\begin{align*}
\Rightarrow j \leq 4. \text{ If } j = 4 \text{ then } a=b \text{ and } \Gamma = E_7
\end{align*}$

If $j=3$, then $3b = 2a+2b$

$\begin{align*}
\Rightarrow b &= 2a \\
c &= 3a \\
\text{and } \Gamma &= E_8
\end{align*}$
Before proving Thm 2,
let's note some general things about the McKay matrix \((M_{ij}) = M\)
for a rep\(n\) \(G \rightarrow \text{GL}(V)\) i.e. \(\chi_i : \chi_j = \sum_{j=0}^l m_{ij} \chi_j\)

PROP: If \(\frac{X_i}{2^k} = \frac{X_j}{2^k}\) for \(i = 0, 1, \ldots, l\) then \(m_{ji} = m_{i+j}\)

(i.e. \(M^T = \text{PMP}\) for a certain non-negative matrix \(P = P^TP\) respecting \(i \rightarrow j\))

proof: \(m_{ji} = \langle X_j \otimes X_i, X_i \rangle_G \frac{1}{|G|} \sum_{g \in G} X_j(g) X_i \chi_i(g) = \frac{1}{|G|} \sum_{g \in G} X_i \chi_i(g) X_j \chi_j(g) = \langle X_i \otimes X_i, X_j \rangle_G = m_{i+j}\)

PROP: Each column \(\begin{bmatrix} X_i(g) \\ X_j(g) \\ \vdots \end{bmatrix}\) of the character table for \(G\)
is an \(M\)-eigenvector, with eigenvalue \(X_i(g)\)

proof: \(X_i(g) X_j(g) = \sum_{j=0}^l m_{ij} X_j(g)\) for \(i = 0, 1, \ldots, l\)

\(\begin{bmatrix} X_i(g) \\ X_j(g) \\ \vdots \end{bmatrix}\)

\(\begin{bmatrix} X_i(g) \\ X_j(g) \\ \vdots \end{bmatrix}\)

\(M\)

COR: The vector \(s = \begin{bmatrix} \chi_i(e) \\ \chi_j(e) \\ \vdots \end{bmatrix}\) of irreducible degrees satisfies \(s^T = M s\)

Also, this \(n\)-eigenspace \(\ker(M - nI) = \text{CS}\) is simple if it is a faithful
rep\(n\) \(G \rightarrow \text{GL}(V)\)

proof: Since the columns \(\begin{bmatrix} \chi_i(g) \\ \vdots \\ \chi_l(g) \end{bmatrix}\) give a basis of eigenvectors,
one only needs to check \(X_i(g) = n\) implies \(g = e\):
\(n = \chi_i(g) = \sum_{i=1}^M \lambda_i; \text{ if } g \text{ has eigenvalues } \lambda_{i_1}, \ldots, \lambda_{i_n} \) on \(V\)
\(\Rightarrow n = \sum_{i=1}^M \lambda_i \leq \sum_{i=1}^M |\lambda_i| = 1 + 1 + \ldots + 1 = n \Rightarrow \lambda_{i_1} = 1 \forall i \Rightarrow g = e\)
**Prop. (Bumslee) If \( G \hookrightarrow GL_n(C) = GL(V) \) is faithful, then every \( G \)-irreducible \( V_i \) appears in some tensor power \( T^m(V) = V \otimes \cdots \otimes V \) (so \( \exists \) a path \( V_0 \to \cdots \to V_i \) in the McKay quiver \( \Gamma \)).

**Proof:** 1st find a vector \( v_0 \in V \) whose \( G \)-orbit is free, i.e. \( g(v_0) \neq v_0 \) unless \( g = e \):

pick any \( v_0 \in V \setminus \bigcup_{g \in G \setminus \{e\}} \ker(g-1) \)

Then consider

\[
\begin{align*}
T^*(V) & \rightarrow \mathfrak{sl}(V) \cong \mathbb{C}[x_{ij}] \rightarrow \{ p : G \rightarrow \mathbb{C} \} \cong \mathbb{C}^G \\
\bigoplus_{m \geq 0} T^m(V) & \rightarrow V^* = \langle x_{ij} \rangle.
\end{align*}
\]

Because of the isomorphism in polynomial interpolation

**Remark:** If \( G \hookrightarrow GL(V) \) has only \( t \) different character values \( \{ \chi(g) \}_{g \in G} \), then Brauer showed every \( V_i \) appears in some \( T^m(V) \) with \( m \leq t-1 \) \((\gamma)\).

Now specialize to \( \begin{align*}
G & \hookrightarrow SL_n(C) = SL(V) \\
\text{and note that } S & = \begin{bmatrix} \chi(e) & \ddots & 0 \\
\ddots & \ddots & \ddots \\
0 & \ddots & \chi(e) \end{bmatrix}
\end{align*} \text{ is additive on the vertices of } \Gamma \text{ by the eigenvalue equation }

2S = MS

2S_i = \sum_{j=0}^{N} m_{ij} S_j.

* \( m_{ij} = m_{ji} \) since \( V_i \cong V^* \) as every \( g \) diagonalizes to \( \begin{bmatrix} \lambda & 0 \\\n0 & \lambda^* \end{bmatrix} \) with \( \lambda \in \mathbb{C}^\times \)

\( \chi_i = \lambda + \lambda^* = \chi_i \).

Hence it only remains for THM 2
to show \( \{ m_{ii} = 0 \}_i \) and \( m_{ij} \in \{0,1\} \).
with nonscalar matrices in its image.

\( m_{ij} = \sum_{g \in G} \chi_i(g) \chi_j(g) \) for any \( G \to SL_2(C) \) with nonscalar matrices in its image.

\[ m_{ij} = \left| \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_j(g) \right| \]

\[ = \left| \frac{1}{|G|} \sqrt{\sum_{g \in G} \left| \chi_i(g) \right|^2 \sum_{g \in G} \left| \chi_j(g) \right|^2} \right| \]

\[ \leq \frac{\sqrt{\sum_{g \in G} \left| \chi_i(g) \right|^2 \sum_{g \in G} \left| \chi_j(g) \right|^2}}{|G|} \]

\[ = \frac{2}{|G|} \sqrt{|G| \cdot |G|} = 2 \]

**Remark:** For subgroups \( G \) of \( SL_2(C) \), containing only scalar matrices forces \( G = \{ I_2, -I_2 \} \) which is type affine \( A_1 \), and in this case, indeed the McKay quiver has two nodes \( 1, 2 \) and \( m_{12} = 2 \).

**Prop:** \( m_{ii} = 0 \) for \( G \to SL_2(C) \) finite

**Proof:** If \( V \) is reducible, then \( g \to \begin{bmatrix} \chi(g) & 0 \\ 0 & \chi(g) \end{bmatrix} \) forces \( G \) cyclic, and of type \( \tilde{A}_m \) as we saw.

If \( V \) is irreducible, then \( 2 = \dim(V) \) divides \( |G| \) by a famous (nontrivial) result about irreducible characters.

So \( G \) contains an element of order 2 by Cauchy's Theorem, and in \( SL_2(C) \) this has to be \( \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I \).

Since \(-I \in Z(G)\), it acts as a scalar \( e \) in any irreducible \( \chi_i \).

So \( \chi_i(g) = e \chi_i(g) \) \( \forall g \in G \), and hence

\[ m_{ii} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g)^2 \]

\[ = \frac{1}{|G|} \sum_{g \in G} (\chi_i(g)^2 + \chi_i(g)^2 + \chi_i(g)^2) \]

\[ = \frac{1}{|G|} \sum_{g \in G} (\chi_i(g)^2 + \chi_i(g)^2 + \chi_i(g)^2) \]

\[ = 0 \]
DEFN: Given a faithful repn $G \to \text{GL}(\mathbb{C})$

define its McKay-Cartan matrix $\tilde{C} := nT - M$

and Cartan matrix $C := \tilde{C} - \{ X_0 \text{ row,  column removed } \}$

and critical group $K(Y) := \text{coker}(\mathbb{Z}^l \overset{\tilde{C}}{\to} \mathbb{Z}^l) = \mathbb{Z}^l/\text{im}C$

PROP: Equivalently,

$\mathbb{Z} \oplus K(Y) = \text{coker}(\mathbb{Z}^{l+1} \overset{\tilde{C}}{\to} \mathbb{Z}^{l+1})$

the torsion part of the cokernel

and $K(Y) = (\mathbb{Z}^3)^l/\text{im}(\tilde{C})$

EXAMPLE: $G = \text{O}_4$ = alternating group in $\Theta_4$

$\to \text{SO}_3(\mathbb{R})$ $(\subset \text{GL}_3(\mathbb{C})$

Character table:

\[
\begin{array}{c|cccc}
\psi & (1,3) & (2,3) & (1,2,3) & (1,3,2) \\
\hline
\chi_0 & 1 & 1 & 1 & 1 \\
\chi_1 & 1 & -1 & 1 & -1 \\
\chi_2 & 1 & -1 & -1 & 1 \\
\chi_3 & 1 & 1 & -1 & -1 \\
\end{array}
\]

$\chi_1 \chi_2 \chi_3 = \chi_3$

$2\chi_1 + \chi_2 + \chi_3 = \chi_3$

$\chi_2 + \chi_3 = \chi_3$

$\chi_0 + \chi_1 + \chi_2 = \chi_3$

$\chi_0 + \chi_1 + \chi_2 + \chi_3 = 3$

\[
M = \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 2
\end{pmatrix}
\]

$C = 3I_4 - M = \begin{pmatrix}
3 & 0 & 0 & -1 \\
0 & 3 & 0 & -1 \\
0 & 0 & 3 & -1 \\
-1 & -1 & -1 & 3
\end{pmatrix}$

$K(Y) = \text{coker}(\mathbb{Z}^3 \overset{C}{\to} \mathbb{Z}^3) = \text{coker}(\mathbb{Z}^3 \overset{\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 2
\end{pmatrix}}{\to} \mathbb{Z}^3) \approx \mathbb{Z}/3\mathbb{Z}$
Some properties of $K(Y)$

- If $G \hookrightarrow SL_2(C)$, not just $GL_n(C)$
  then $\exists$ a surjection $K(Y) \to G := \text{Hom}(G, C^\times)$ via
  \[
  G \cong \mathfrak{g} / [\mathfrak{g}, \mathfrak{g}]
  \]
  also a submanifold of $G_y$.

Example: For $U_4 \hookrightarrow SO_3(\mathbb{R}) \subset SL_3(C)$ above,

\[
K(Y) \cong \mathbb{Z}/3\mathbb{Z}
\]

and $\mathfrak{g}_{ab} \cong \mathbb{Z}/3\mathbb{Z}$

Example: For McKay's original setting, where $G \hookrightarrow SL_2(C)$

one always has an isomorphism $K(Y) \cong \mathfrak{g}_{ab}$

but also

\[
\text{coker}(C) \cong \pi_1 \text{ fundamental group of } \Phi
\]

usual Cartan matrix

expressing the simple roots $\{\alpha_1, \ldots, \alpha_n\}$

in the basis of fundamental weights $\{\lambda_1, \ldots, \lambda_n\}$

\[
C = \begin{bmatrix}
  2 & 1 & -1 & -1 \\
  1 & 2 & -1 & -1 \\
  -1 & -1 & 2 & -1 \\
  -1 & -1 & -1 & 2
\end{bmatrix}
\]

$\mathfrak{g}_{ab} = \{\mathbb{Z}2\mathbb{Z} \text{ if } G+y, \text{ even} \}

\{\mathbb{Z}/2\mathbb{Z} \text{ if } G+y, \text{ odd} \}$

- $\mathbb{Z} + K(Y) = \text{coker}(C)$ is a naturally a ring

and $K(Y)$ itself an ideal in this ring:

PROP: Considering the representation ring $R(G) := \mathbb{Z}[\text{Im}(G)]$ having $\{e_1, e_2, \ldots, e_n\} \to \mathbb{Z}$ as $\mathbb{Z}$-basis $\chi \mapsto \langle \chi \rangle$

\[
e_i \cdot e_j = \sum_{k=0}^{i} c_{ijk} e_k
\]

$G \cong GL_n(C)$ has

$\text{coker}(C) \cong R(Y) = R(G) / (n-e_y)$

where $e_y = \sum_{k=0}^{n} c_{yjk} e_k$

\[
\text{proof: } n-e_y \text{ acts on the } \mathbb{Z}\text{-basis } \{e_1, e_2, \ldots, e_n\} \text{ for } R(G)
\]

via the matrix $C$.}

\]
EXAMPLE: \( R(\mathcal{G}) \cong \mathbb{Z}\{x_0, x_1, x_2, x_3\} \cong \mathbb{Z}^4 \)

\[ \cong \mathbb{Z}[x, y]/(x^2 - 1, xy - y, y^2 - (2y + x + x^2)) \]

So \( \text{Gal}(\mathcal{G}) \to S_3(\mathbb{R}) \)

has \( R(Y) = \mathbb{Z}[x, y]/(x^2 - 1, xy - y, y^2 - (2y + x + x^2), (x - y)) \)

\[ \cong \mathbb{Z}[x]/(x^2 - 1, 3(x - 1), 9 - (6 + 2 + x + x^2)) \]

\[ \cong \mathbb{Z}[x]/(3(x - 1), (x - 1)^2) \]

\[ \cong \mathbb{Z}[u]/(3u, u^2) = \mathbb{Z}: 1 \oplus (\mathbb{Z}/3\mathbb{Z})u \]

\[ = \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} = \text{ker} (\mathcal{C}) \]

\[ K(Y) \]

- If \( G \) is abelian, so \( G \cong \hat{G} = \text{Irr}(G) \)

where any map \( G \to \text{Gl}_n(\mathbb{C}) \) has \( K(Y) \cong \text{usual digraph critical group} \)

for the Cayley digraph of \((\hat{G}, \{q_{\mathcal{G}} \text{ copies } X_k \})\)

EXAMPLE: \( G = (\mathbb{Z}/2\mathbb{Z})^n \to \text{Gl}_n(\mathbb{C}) \)

\[ \{q_{\mathcal{G}} \text{ copies } X_k \} \]

has \( K(Y) = K(\mathcal{Q}_n) \)

\[ \mathbb{Z} : K(Y) \cong \mathbb{Z}[x_1, \ldots, x_n]/(x_1^2 - 1, \ldots, x_n^2 - 1, n(x_1 \cdots x_n)) \]

Q: Does this help to understand the 2-primary structure of \( K(\mathcal{Q}_n) \)?

(Hua Bai computed the p-primary structure for odd \( p \), which is much easier)

Q: What does \( K(\mathcal{N}) \) for \( G_n \to \text{Gl}_n(\mathbb{C}) \) look like?
T. Douvropoulos  Geometric McKay Correspondence  2/22/2016

Kepler: Saturn Jupiter Mars Earth Venus Mercury
  cube  tetrahedron  dodecahedron  icosahedron  octahedron

Algebraic McKay correspondence
\[ \Gamma \subset \text{SL}(2,\mathbb{C}) \quad \longrightarrow \quad \text{extended Dynkin diagram} \]
finite irreps \[ \longrightarrow \quad \text{vertices} \]

Geometric McKay correspondence
\[ \Gamma \subset \text{SL}(2,\mathbb{C}) \quad \longrightarrow \quad \mathbb{C}^2/\Gamma \quad \longrightarrow \quad (\text{nonextended}) \quad \text{Dynkin diagram} \]
finite unique singularity \[ \longrightarrow \quad \text{vertices} \]
embedded in \( \mathbb{C}^3 \)

irred. components of the exceptional divisor \[ \longrightarrow \quad \text{vertices} \]

Two pictures:

\[ A_1: \quad C = \{ \pm 1 \} \subset \text{SL}(2,\mathbb{C}) \]
defining eqn. of \( C/\Gamma \quad x^2+y^2=0 \)
(equiv. to \( x^2+y^2-z^2=0 \))
\[ \pi^{-1}(o) = \quad \text{Dynkin diagram} \]

\[ A_1 = \quad \text{Dynkin diagram} \]

\[ D_4: \quad \text{defining eqn. of } C^2/\Gamma \quad x(y^2-x^2)+z^2=0 \]
\[ \pi \]
\[ E = \pi^*(o) = \begin{array}{c}
\text{Intersection diagram}
\end{array} \rightarrow \begin{array}{c}
D_4 \text{ Dynkin diagram}
\end{array} \]

\[ \text{8.1 Invariant theory} \]

Finite group \( G \) acts via linear transformation on \( V = C^n \),
also acts on \( \text{C}[V]^G = C[x_1, ..., x_n] \)
via \( g.f = f(g.x) \)
Consider invariant subalgebra \( \text{C}[V]^G = \{ f \in \text{C}[V] : f(x) = f(g.x) \} \)

**FACT (Noether-Hilbert)** \( \text{C}[V]^G \) is generated by finitely many polynomials

**FACT (Klein-Du Val)** \( \Gamma \subset \text{SL}(2,C) \Rightarrow \text{C}[V]^G \) is gen\( \ell \)ed by exactly 3 polynomials

\( \Gamma \subset \text{SL}(2,C) \Rightarrow \text{C}[V]^G \) is gen\( \ell \)ed by exactly 3 polynomials

**Example 1:** \( \Gamma = \text{C}_n \) gen\( \ell \)ed by \( \begin{bmatrix} e^{\frac{2\pi i}{n}} & 0 \\
0 & e^{-\frac{2\pi i}{n}} \end{bmatrix} \)

\( f(x, y) = (f.x, f.y) \)

\( \text{Invariants: } x^n, y^n, xy \quad f_1^n - f_2 f_3 = 0 \)

\( \text{G.I.T. say } \text{C}[V]^\Gamma \text{ and } V/\Gamma \text{ are deeply related} \)

Indeed, \( \text{C}[f_1, f_2, f_3] \hookrightarrow \text{C}[x, y] \) induces a map

\( \begin{bmatrix}
(f(x), f(x), f(x)) \\
(x^n, y^n, xy)
\end{bmatrix} \)

\( \begin{bmatrix}
(x, y)
\end{bmatrix} \)
This map $\mathbb{C}^2 \to \mathbb{C}^3$ realizes $\mathbb{C}^2/\Gamma$ as a topological quotient.

**Example:**

$\mathbb{C}^2 \to \mathbb{C}^3$  \quad $\Gamma = \mathbb{G}_m$

$(xy) \mapsto (xy, x^n, y^n)$

Image in $\mathbb{C}^3$ with coords $(x, y, z)$ is cut out by $x^n - yz$

§2 Resolution of singularity

Recall

Resolving the singularity means finding $S$ and $\tilde{S} \xrightarrow{\pi} S$

such that $\tilde{S} \setminus \pi(0) \simeq S \setminus \{0\}$

Consider $B(0, \mathbb{C}^n) = \{(v, L) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid v \in L\}$

$n = 3$: $\{((x, y, z), [s : t : u]) : xt = su, xu = sz, tz = uy\}$

$\beta(\mathbb{C}^3/\{0\}) \cong \mathbb{C}^3 \setminus \{0\}$

$\beta: \mathbb{C}^3 \cong \mathbb{P}^{n-1}$

Given $S \subset \mathbb{C}^n$ singular at $0$,

then the blow-up of $0 \in S$ is $B(0, S) = \beta'(S \setminus \{0\})$

**Exceptional divisor** $E = B(0, S) \cap \beta'(0)$
Resolution of type $A_n$, $\Gamma = C_{N+1}$

Singularity has equation $x^{N+1} - yz = 0$ in $C^3$, equivalent to $x^{N+1} - y^2 + z^2 = 0$.

In $\tilde{U}_1$, $x^{N+1} + (xt)^2 + (xu)^2 = 0$.

$\Rightarrow x^2(x^{N+1} + t^2 + u^2) = 0$.

If $x = 0$, get all of $\{(0,0,0), [1:t:u]\}$, which corresponds to $B(0) \cap \tilde{U}_1$.

$x^{N+1} + t^2 + u^2 = 0$ should give me $\beta^{-1}(0 \times 0) \cap \tilde{U}_1$.

$N = 1 \Rightarrow 1 + t^2 + u^2 = 0$.

$E$ is given in $\tilde{U}_1$ by $[1:t:s]$ such that $1 + t^2 + s^2 = 0$.

(At $\tilde{U}_1$, would get $s^3(y)^{N+1} + 1 + u^2 = 0$, and $N=1 \Rightarrow s^2 + 1 + u^2 = 0$).

$\Rightarrow E_1$ is just $s^2 + t^2 + u^2 = 0$, a copy of $P^1 \subset P^2$.

$N > 1 \Rightarrow E \cap \tilde{U}_1 = \{(t^2 + u^2 = 0)\} = \{[1:a : \pm a] : a \in C\}$

$E \cap \tilde{U}_2 = \{s^{N+1} + u^2 + 0 \ \text{or} \ y = 0\} = \{[s:1:u] : u^2 = 0\}$

$= \{[b:1:tc]\}$.
So $E = \{ [a:a:tz] \}$ and $\{ [b:1:tz] \}$

If $b = \frac{1}{a}$, $[a:a:tz] = [b:1:tz]$

$E$ has 2 lines, $\times$

but we have not yet resolved the singularity if $n \geq 2$

since the equation $z^{n+1} + t^2 + u^2 = 0$ is still singular.

You proceed inductively, and keep blowing up to get

\[
\begin{array}{c}
\times \\
\times \\
\times \\
\times \\
\times \\
\rightarrow \text{type An Dynkin diagram}
\end{array}
\]

A schematic picture:


- hard to access!

\[
\begin{align*}
\text{Du Val Singularities} & \rightarrow \text{Finite subgroups of SL}_2(\mathbb{C}) \\
& \rightarrow \text{McKay Correspondence} \\
& \rightarrow \text{Dynkin diagrams} \\
& \rightarrow \text{Coxeter relations} \\
& \rightarrow \text{Reflection groups}
\end{align*}
\]