(1)

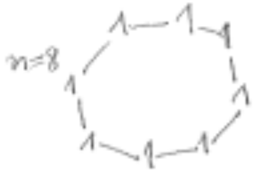
McKay Correspondence Feb. 1, 2016

(Refs: Yi Sun, Sternberg "Fin. subgps of SU_2, ...")

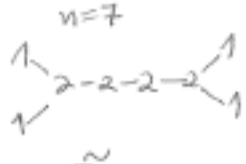
Let's classify finite subgroups $G \hookrightarrow SL_2(\mathbb{C}) = SL(V)$, $V = \mathbb{C}^2$ by proving two things ...

(Old) THM: Γ a connected finite graph has a vertex-labeling $f: V \to \{1,2,\ldots\}$ which is $\boxed{\text{additive } 2f(v) = \sum_{\substack{w \in V: \\ w - v}} f(w)}$

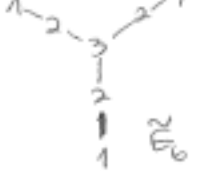
$\iff$ Γ is an $\boxed{\text{ADE (simply-laced affine) Dynkin diagram}}$ labeled as follows, up to scaling:



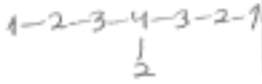
$\tilde{A}_n$, $n \geq 2$



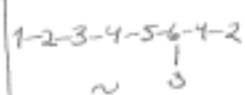
$\tilde{D}_n$, $n \geq 4$



$\tilde{E}_6$



$\tilde{E}_7$



$\tilde{E}_8$

(McKay ~1980) THM: For a finite subgroup $G \hookrightarrow SL_2(\mathbb{C})$, its McKay quiver Γ having vertex set $\underset{\substack{\text{irred.} \\ \mathbb{C}\ G\text{-characters}}}{\mathrm{Irr}(G)} = \{\underset{\mathbb{1}_G}{\overset{\shortparallel}{\chi_0}}, \chi_1, \ldots, \chi_\ell\}$, and m_{ij} arcs $\chi_i \overset{m_{ij}}{\rightrightarrows} \chi_j$ if $\boxed{\chi_i \otimes \chi_V = \sum_{j=0}^{\ell} m_{ij} \chi_j}$

(except m_{12}=2 in affine A_1)

is always connected, loopless, and $m_{ij} = m_{ji} \in \{0, 1\}$ (so it corresponds to a graph Γ)

χ_i $\cancel{\circlearrowleft}$ $\chi_i \rightleftarrows \chi_j$

and $f(\chi_i) = \deg(\chi_i) := \chi_i(e)$ is additive, so Γ is ADE as above.

EXAMPLE: $G = \mathbb{Z}/n\mathbb{Z} \hookrightarrow SL_2(\mathbb{C}) = SL(V)$
$\{e, g, g^2, \ldots, g^{n-1}\}$ $g \longmapsto \begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$
$\zeta := e^{2\pi i/n}$

has $\mathrm{Irr}(G) = \overbrace{\{\chi_0, \chi_1, \chi_2, \ldots, \chi_{n-1}\}}^{\text{all degree } 1}$
$\chi_0 = \mathbb{1}_G$ where $\chi_1(g) = \zeta^1$
$\chi_i(g) = \zeta^i$

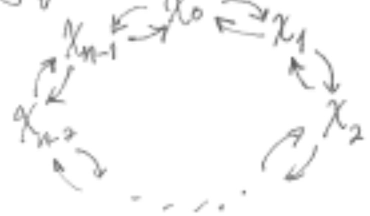
$\chi_V = \chi_1 + \chi_{n-1}$

$\chi_i \otimes \chi_V = \chi_{i-1} + \chi_{i+1}$

subscripts mod n

and the McKay quiver for V is

$\tilde{A}_n =$



proof of THM 1:

- If Γ has a cycle, it is that cycle:

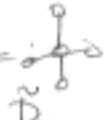
$2f(v_i) = v_{i-1} + v_{i+1} + \dots$ mod k

$\geq v_{i-1} + v_{i+1}$

$\Rightarrow 2\sum_{i=0}^{k-1} f(v_i) \geq \sum_{i=0}^{k-1} f(v_i) + \sum_{i=0}^{k-1} f(v_i))$

$\Rightarrow$ equality everywhere, so no neighbors outside the cycle.

So WLOG, Γ is a tree.

- There are no vertices of degree ≥ 5, and if one has degree 4, then $\Gamma =$  $\tilde{D}_4$:

$\Rightarrow 2f(v) = f(v_1) + \dots + f(v_5) + \dots$

$\geq \sum_{i=1}^{d} f(v_i)$

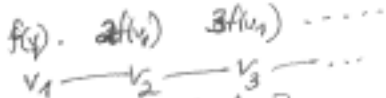
and $2\sum_{i=1}^{d} f(v_i) = 5f(v) + \dots \geq 5f(v)$

$\rightarrow \frac{d}{4} f(v) \leq \frac{1}{2}\sum_{i=1}^{d} f(v_i) \leq f(v)$

$\Rightarrow$ contradiction unless $d \leq 4$, and if $d=4$ it forces equality everywhere, so no other vertices.

So WLOG, Γ has max degree ≤ 3.

- There exist vertices of degree 3, else Γ is a path, forcing this labeling:



Contradiction to finiteness!

- If there are 2 vertices of degree 3, connected by a path, then

there is nothing else, i.e. it is $\tilde{D}_n$:

by this calculation:

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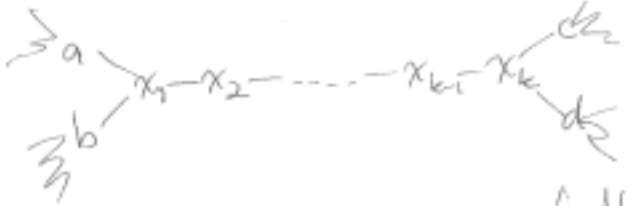


$$\left.\begin{array}{l} 2a \geq x_1 \\ 2b \geq x_1 \\ \hline 2c \geq x_k \\ 2d \geq x_k \end{array}\right\} \Rightarrow 2(a+b+c+d) \geq 2(x_1+x_k)$$
$$a+b+c+d \underset{(*)}{\geq} x_1+x_k$$

Also
$$\begin{aligned} 2x_1 &= a+b+x_2 \\ 2x_2 &= x_1+x_3 \\ 2x_3 &= x_2+x_4 \\ 2x_4 &= x_3+x_5 \\ &\vdots \\ 2x_{k-1} &= x_{k-2}+x_k \\ 2x_k &= c+d+x_{k-1} \end{aligned}$$
$$2\sum_{i=1}^{k} x_i = a+b+c+d+2\sum_{i=1}^{k} x_i - (x_1+x_k)$$
$$\Rightarrow x_1+x_k = a+b+c+d \underset{(*)}{\geq} x_1+x_k$$

forces equality, so no other vertices

So WLOG, Γ has only one deg 3 vertex.

It looks like this:

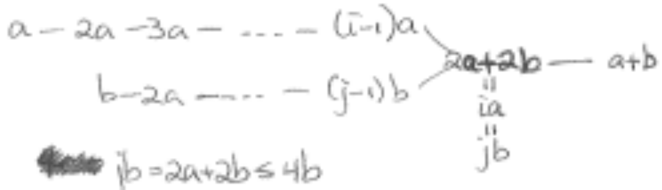
with $a \leq b \leq c$
and $i \geq j \geq k$

Then $2x = 3x - (a+b+c)$

i.e. $\overset{kc}{\overset{\shortparallel}{x}} = a+b+c \leq 3c \Rightarrow k \leq 3$

- If $k=3$, then $x = 3c = a+b+c$ forces $a=b=c$ and $\Gamma = \tilde{E}_6$

- If $k=2$, then $c=a+b$ and $\Gamma =$



and $jb = 2a+2b \leq 4b$

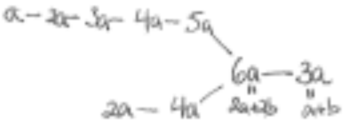
$\Rightarrow j \leq 4$. If $j=4$ then $a=b$ and $\Gamma = \tilde{E}_7$

If $j=3$, then $3b = jb = 2a+2b$

so $b = 2a$

$c = 3a$

and $\Gamma = \tilde{E}_8$



Before proving THM 2,
let's note some general things about the McKay matrix $(m_{ij}) =: M$
for a rep'n $G \longrightarrow GL_n(\mathbb{C}) = GL(V)$ i.e. $\chi_i \otimes \chi_V = \sum_{j=0}^{\ell} m_{ij} \chi_j$

<u>PROP:</u> If $\chi_{i^*} = \overline{\chi_i}$ for $i = 0, 1, \ldots, \ell$ then $m_{ji} = m_{i^* j^*}$
(i.e. $M^T = PMP$ for a certain involutive perm. matrix $P = P^T = P^{-1}$ enacting $i \mapsto i^*$)

<u>proof:</u>
$$m_{ji} = \langle \chi_j \otimes \chi_V, \chi_i \rangle_G = \frac{1}{|G|} \sum_{g \in G} \chi_j(g) \chi_V(g) \chi_i(g^{-1})$$
$$= \frac{1}{|G|} \sum_{g \in G} \chi_{j^*}(g^{-1}) \chi_V(g) \chi_{i^*}(g) = \underbrace{\langle \chi_{i^*} \otimes \chi_V, \chi_{j^*} \rangle}_{= m_{i^* j^*}} \quad ■$$

<u>PROP:</u> Each column $\begin{bmatrix} \chi_0(g) \\ \chi_1(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$ of the character table for G
is an M-eigenvector, with eigenvalue $\chi_V(g)$

<u>proof:</u>
$$\underbrace{\chi_i(g)\, \chi_V(g)}_{i^{th} \text{ entry of } \chi_V(g) \begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}} = \underbrace{\sum_{j=0}^{\ell} m_{ij} \chi_j(g)}_{i^{th} \text{ entry of } M \begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}} \quad \text{for } i = 0, 1, \ldots, \ell \quad ■$$

<u>COR:</u> The vector $\delta = \begin{bmatrix} \chi_0(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix}$ of irreducible degrees satisfies $n\delta = M\delta$

Also, this n-eigenspace $\ker(M - n \cdot I) = \mathbb{C}\delta$ is <u>simple</u> if it is a <u>faithful</u> rep'n $G \hookrightarrow GL_n(\mathbb{C})$

<u>proof:</u> Since the columns $\begin{bmatrix} \chi_0(g) \\ \vdots \\ \chi_\ell(g) \end{bmatrix}$ give a basis of eigenvectors,

one only needs to check $\chi_V(g) = n$ implies $g = e$:
$$n = \chi_V(g) = \sum_{i=1}^{n} \lambda_i \quad \text{if } g \text{ has eigenvalues } \lambda_1, \ldots, \lambda_n \text{ on } V$$
$$\Rightarrow n = \left| \sum_{i=1}^{n} \lambda_i \right| \underset{\uparrow \text{ Cauchy-Schwarz}}{\le} \sum_{i=1}^{n} |\lambda_i| = \underbrace{1+1+\ldots+1}_{n \text{ times}} = n \quad \Rightarrow \lambda_i = 1 \ \forall i \Rightarrow g \text{ acts as } 1_V$$
$$\underset{\text{faithful}}{\Rightarrow} g = e \quad ■$$

PROP (Burnside) If $\underset{\text{finite}}{G} \hookrightarrow GL_n(\mathbb{C}) = GL(V)$ is faithful, then every G-irreducible χ_i appears in some tensor power $T^m(V) = \underbrace{V\otimes\cdots\otimes V}_{m \text{ times}}$,

(so $\exists$ a path $\chi_0 \underbrace{\to \cdots \to}_{m \text{ steps}} \chi_i$ in the McKay quiver Γ)

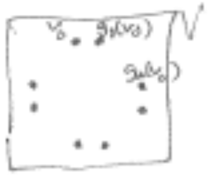
proof: 1st find a vector $v_0 \in V$ whose G-orbit is free, i.e. $g(v_0) \neq v_0$ unless $g = e$:

$$\text{pick any } v_0 \in V \setminus \underbrace{\bigcup_{g \in G\setminus\{e\}} \underbrace{\ker(g - 1_V)}_{\text{a proper subspace} \subsetneq V}}_{\text{a finite union of proper subspaces} \subsetneq V}$$

Then consider

$$\begin{tikzcd}
T^\bullet(V^*) \arrow[r, two heads] \arrow[d, equal] & S^\bullet(V^*) \cong \mathbb{C}[x_1,\dots,x_n] \arrow[rr, two heads] & & \left\{\begin{matrix}\text{functions}\\ \varphi: G \to \mathbb{C}\end{matrix}\right\} \cong \underset{\substack{\text{regular}\\ \text{rep'n}}}{\mathbb{C}G} \supseteq \chi_i^* \\
\bigoplus_{m\geq 0} T^m(V^*) & V^* = \mathbb{C}x_1 + \dots + \mathbb{C}x_n \arrow[u] & f(x) \arrow[r, mapsto] & \left(\varphi(g) = f(g(v_0))\right)
\end{tikzcd}$$

surjectivity is an exercise in polynomial interpolation



$\blacksquare$

RMK: If $\underset{\text{finite}}{G} \hookrightarrow GL(V)$ has only t different character values $\{\chi_V(g)\}_{g\in G}$ then Brauer showed every χ_i appears in some $T^m(V)$ with $m \leq t-1$ (?).

Now specialize to $\underset{\text{finite}}{G} \hookrightarrow SL_2(\mathbb{C}) = SL(V)$

and note that
- $\delta = \begin{bmatrix} \chi_0(e) \\ \chi_1(e) \\ \vdots \\ \chi_\ell(e) \end{bmatrix}$ is additive on the vertices of Γ by the eigenvalue equation $2\delta = M\delta$
$$2\delta_i = \sum_{j=0}^{\ell} m_{ij}\delta_j$$
- $m_{ij} = m_{ji}$ since $V \cong V^*$ as every g diagonalizes to $\begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$ with $\lambda^{-1} = \bar{\lambda} \in \mathbb{C}^\times$
so $\chi_V = \lambda + \lambda^{-1} = \chi_{V^*}$

Hence it only remains for THM 2 to show $\left\{\begin{matrix} m_{ii} = 0 \\ \text{and} \\ m_{ij} \in \{0,1\} \end{matrix}\right.$

(6)

[redacted]

PROP: $m_{ij} \in \{0,1\}$ for $\underset{\text{finite}}{G} \to GL_2(\mathbb{C})$ any ~~nontrivial~~ rep'n with nonscalar matrices in its image.

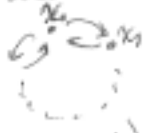
proof:
$$m_{ij} = |m_{ij}| = \left| \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_V(g) \overline{\chi_j(g)} \right|$$
$$\leq \frac{1}{|G|} \sqrt{\sum_{g \in G} |\chi_i(g)|^2 \underbrace{|\chi_V(g)|^2}_{\substack{\leq 2^2 \\ \text{since } \chi_V(g) = \lambda + \lambda^{-1} \\ |\lambda| = 1}}} \cdot \sqrt{\sum_{g \in G} |\chi_j(g)|^2}$$

since it's a ~~nontrivial~~ rep'n with nonscalar matrices in its image.
$$< \frac{2}{|G|} \sqrt{\sum_{g \in G} |\chi_i(g)|^2} \sqrt{\sum_{g \in G} |\chi_j(g)|^2} = \frac{2}{|G|} \sqrt{|G|} \cdot \sqrt{|G|} = 2 \quad ■$$

REMARK: For subgroups G of SL_2(C), containing only scalar matrices forces G={ I_2, -I_2 } which is type affine A_1, and in this case, indeed the McKay quiver has two nodes 1,2 and m_{12}=2.

PROP: $m_{ii} = 0$ for $\underset{\text{finite}}{G} \hookrightarrow SL_2(\mathbb{C}) = SL(V)$

proof: If V is reducible, then $g \longmapsto \left[\begin{array}{c|c} \lambda(g) & 0 \\ \hline 0 & \lambda(g)^{-1} \end{array}\right]$ forcing G cyclic, and of type $\tilde{A}_n$ as we saw



If V is irreducible, then $2 = \dim_{\mathbb{C}}(V)$ divides $|G|$ by a famous (nontrivial) result about irred. characters

so G contains an element of order 2 by Cauchy's Thm., and in $SL_2(\mathbb{C})$ this has to be $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I$.

Since $-I \in Z(G)$, it acts as a scalar $\underbrace{\,}_{\epsilon = +1 \text{ or } -1}$ in any irreducible χ_i

so $\chi_i(-g) = \epsilon \chi_i(g) \quad \forall g \in G$, and hence

$$m_{ii} = \frac{1}{|G|} \sum_{g \in G} \chi_i(g) \chi_V(g) \overline{\chi_i(g)} = \frac{1}{|G|} \sum_{g \in G} |\chi_i(g)|^2 \chi_V(g)$$
$$2m_{ii} = \frac{1}{|G|} \sum_{g \in G} \left(\chi_V(g) |\chi_i(g)|^2 + \chi_V(-g) |\chi_i(-g)|^2 \right)$$
$$= \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \underbrace{\left(|\chi_i(g)|^2 - \underbrace{|\epsilon \chi_i(g)|^2}_{|\chi_i(g)|^2} \right)}_{=0} \quad ■$$

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DEF'N: Given a faithful rep'n $G \xhookrightarrow{\gamma} Gl_n(\mathbb{C})$

define its McKay-Cartan matrix $\tilde{C} := nI_{l+1} - M$

and Cartan matrix $C := \tilde{C} - \left\{ \begin{matrix} \chi_0 \text{ row,} \\ \text{column} \\ \text{removed} \end{matrix} \right\}$

and critical group $K(\gamma) := \operatorname{coker}(\mathbb{Z}^l \xrightarrow{C} \mathbb{Z}^l)$

$= \mathbb{Z}^l / \operatorname{im} C$

$$\tilde{C} = \begin{array}{c|cccc} & \chi_0 & \chi_1 & \cdots & \chi_l \\ \hline \chi_0 & * & * & \cdots & * \\ \chi_1 & * & & & \\ \vdots & \vdots & & C & \\ \chi_l & * & & & \end{array}$$

PROP: Equivalently,

(see PROP 2.19 in paper with Benkart & Klivans)

$$\mathbb{Z} \oplus \underbrace{K(\gamma)}_{\substack{\text{the}\\ \text{torsion}\\ \text{part of}\\ \text{the cokernel}}} \cong \operatorname{coker}(\mathbb{Z}^{l+1} \xrightarrow{\tilde{C}} \mathbb{Z}^{l+1})$$

and $K(\gamma) \cong \left(\delta^{(e)}\right)^{\perp} / \operatorname{im}(\tilde{C})$

EXAMPLE: $G = \mathfrak{A}_4$ = alternating group in $\mathfrak{S}_4$ $\xhookrightarrow{\gamma = \text{rotational symmetries of tetrahedron}} SO_3(\mathbb{R}) \ (\subset Gl_3(\mathbb{C}))$

Character table:

$\omega := e^{\frac{2\pi i}{3}}$

	e	(123)	(132)	$(12)(34)$
$1_G = \chi_0$	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
$\left(\underset{\text{for } i=0,1,2}{\chi_i \chi_3 =}\right) \chi_\gamma = \chi_3$	3	0	0	-1
$2\chi_3 + \chi_0 + \chi_1 + \chi_2 = \chi_3^2$	9	0	0	1

$$M = (m_{ij}) = \begin{array}{c|cccc} & \chi_0 & \chi_1 & \chi_2 & \chi_3 \\ \hline \chi_0 & 0 & 0 & 0 & 1 \\ \chi_1 & 0 & 0 & 0 & 1 \\ \chi_2 & 0 & 0 & 0 & 1 \\ \chi_3 & 1 & 1 & 1 & 2 \end{array}$$

$$\tilde{C} = 3I_4 - M = \begin{array}{c|cccc} & \chi_0 & & & \\ \hline \chi_0 & 3 & 0 & 0 & -1 \\ & 0 & 3 & 0 & -1 \\ & 0 & 0 & 3 & -1 \\ & -1 & -1 & -1 & 1 \end{array}$$

$$C = \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

$\det \begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix} = 3$

$$K(\gamma) = \operatorname{coker}\left(\mathbb{Z}^3 \xrightarrow{\overset{\begin{bmatrix} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{bmatrix}}{\overset{\parallel}{C}}} \mathbb{Z}^3\right)$$

$$= \operatorname{coker}\left(\mathbb{Z}^3 \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}} \mathbb{Z}^3\right)$$

$$\cong \mathbb{Z}/3\mathbb{Z}$$

Some properties of $K(\gamma)$

- If $G \overset{\gamma}{\hookrightarrow} SL_n(\mathbb{C})$, not just $GL_n(\mathbb{C})$

  then $\exists$ a surjection $K(\gamma) \xrightarrow{\pi} \hat{G} := \mathrm{Hom}(G, \mathbb{C}^\times) \cong \underbrace{G^{ab} = G/[G,G]}_{\text{abelianization of } G}$

EXAMPLE: For $Cl_4 \overset{\gamma}{\hookrightarrow} SO_3(\mathbb{R}) \subset SL_3(\mathbb{C})$ above

$$K(\gamma) \cong \mathbb{Z}/3\mathbb{Z}$$
$$\pi \Big\downarrow \wr$$
and $Cl_4^{ab} \cong \mathbb{Z}/3\mathbb{Z}$

EXAMPLE: For McKay's original setting, where $\underset{\text{finite}}{G} \overset{\gamma}{\hookrightarrow} SL_2(\mathbb{C})$

one always has an isomorphism

$$K(\gamma) \cong G^{ab}$$

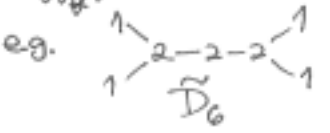
but also

$$\begin{array}{c} \| \\ \mathrm{coker}(\tilde{C}) \end{array} \cong \text{weight lattice } P(\Phi) \Big/ \underset{\text{root lattice}}{Q(\Phi)} \cong \pi_1\left(\begin{array}{c}\text{compact}\\ \text{adjoint}\\ \text{form}\\ \text{of the}\\ \text{semisimple}\\ \text{Lie}\\ \text{group}\end{array}\right)$$

"fundamental group" of Φ

C = usual Cartan matrix expressing the simple roots $\{\alpha_1, \ldots, \alpha_n\}$ in the basis of fund'l weights $\{\lambda_1, \ldots, \lambda_n\}$

e.g.



$$\tilde{C} = \begin{bmatrix} 2 & & -1 & & & & \\ & 2 & -1 & & & & \\ -1 & -1 & 2 & -1 & & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & -1 \\ & & & & -1 & 2 & \\ & & & & -1 & & 2 \end{bmatrix}$$

$$G^{ab} \cong \begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } G \leftrightarrow \tilde{D}_n \ n \text{ even} \\ \mathbb{Z}/4\mathbb{Z} & \text{if } G \leftrightarrow \tilde{D}_n \ n \text{ odd} \end{cases}$$

- $\mathbb{Z} \oplus K(\gamma) = \mathrm{coker}(\tilde{C})$ is a naturally a ring

  and $K(\gamma)$ itself an ideal in this ring:

PROP: Considering the representation ring $R(G) := \mathbb{Z}^{\mathrm{Irr}(G)}$ having $\underset{\text{as } \mathbb{Z}\text{-basis}}{\{\overset{1}{e_0}, e_1, \ldots, e_\ell\}} \xrightarrow{\epsilon} \mathbb{Z}$, $\chi \mapsto \underset{\deg \chi}{\chi(e)}$

and $e_i \cdot e_j = \sum_{k=0}^{\ell} c_{ij}^k e_k$ if $\chi_i \otimes \chi_j = \sum_{k=0}^{\ell} c_{ij}^k e_k$,

then $\overset{G \overset{\gamma}{\hookrightarrow} GL_n(\mathbb{C}) \text{ has}}{\gamma}$ $\mathrm{coker}(\tilde{C}) \cong R(\gamma) := R(G) \big/ (n - e_\gamma)$ where $e_\gamma = \sum_{k=0}^{\ell} c_k e_k$ if $\chi_\gamma = \sum_{k=0}^{\ell} c_k \chi_k$

proof: $n - e_\gamma$ acts ~~on~~ the $\mathbb{Z}$-basis $\{e_0, e_1, \ldots, e_\ell\}$ ~~for~~ $R(G)$ via the matrix $\tilde{C}$ ∎

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<u>EXAMPLE:</u> $R(\underset{\mathfrak{S}_4}{G}) \cong \mathbb{Z}\{\overset{1}{\chi_0}, \overset{x}{\chi_1}, \underset{\chi_1 \otimes \chi_1}{\overset{x^2}{\chi_2}}, \overset{y}{\chi_3}\} \cong \mathbb{Z}^4$

$\cong \mathbb{Z}[x,y]/(\underline{x}^3-1, \underline{xy}-y, \underline{y}^2-(2y+1+x+x^2))$

So $\mathfrak{S}_4 \xhookrightarrow{\gamma} SO_3(\mathbb{R})$

has $R(\gamma) = \mathbb{Z}[x,y]/((x^3-1, xy-y, y^2-(2y+1+x+x^2), \boxed{3-y}))$

$\cong \mathbb{Z}[x]/(x^3-1, 3(x-1), \underbrace{9-(6+1+x+x^2)}_{-(x^2+x-2)})$

$\cong \mathbb{Z}[x]/(3(x-1), (x-1)^2)$

$\cong \mathbb{Z}[u]/(3u, u^2) = \mathbb{Z}\cdot 1 \oplus (\mathbb{Z}/3\mathbb{Z})u$

$= \mathbb{Z} \oplus \underbrace{\mathbb{Z}/3\mathbb{Z}}_{K(\gamma)} = \mathrm{coker}(\tilde{C})$

- If G is <u>abelian</u>, so $G \cong \hat{G} = \mathrm{Irr}(G)$ $\quad \tilde{C}$ = <u>usual</u> digraph <u>Laplacian</u>
  then any rep'n $G \xhookrightarrow[\sum a_k \chi_k]{\gamma} GL_n(\mathbb{C})$ has $K(\gamma) \cong$ <u>usual</u> digraph <u>critical</u> group
  for the Cayley digraph of $(\hat{G}, \{a_k \text{ copies } \chi_k \text{ for } k=0,1,\dots,\ell\})$

<u>EXAMPLE:</u> $G = (\mathbb{Z}/2\mathbb{Z})^n \xhookrightarrow{\gamma} GL_n(\mathbb{C})$

$= \langle g_1, \dots, g_n : g_i^2 = e \rangle$

$g_i \longmapsto \begin{bmatrix} 1 & & & & & 0 \\ & \ddots & & & & \\ & & 1 & & & \\ & & & \boxed{-1} & & \\ & & & & 1 & \\ 0 & & & & & \ddots \; 1 \end{bmatrix}$

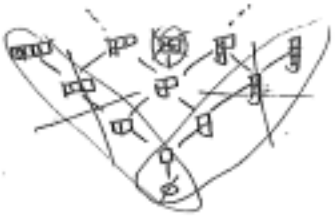
has $K(\gamma) = K(\overset{Q_n}{n\text{-cube}})$

so $\mathbb{Z} \oplus K(\gamma) \cong \mathbb{Z}[x_1, \dots, x_n]/(x_1^2-1, \dots, x_n^2-1, n-(x_1+\dots+x_n))$

<u>Q</u>: Does this help to understand the 2-<u>primary</u> structure of $K(Q_n)$?
(Hua Bai computed the p-primary structure for odd p, which is <u>much easier</u>.)

<u>Q</u>: What does $K(\chi^\lambda)$ for $\mathfrak{S}_n \xhookrightarrow{\chi^\lambda} GL_{f^\lambda}(\mathbb{C})$ look like?

$\lambda \neq \square\square\square\square, \begin{smallmatrix}\square\\\square\\\square\\\square\end{smallmatrix}, \underset{n=4}{\boxplus}$



(1)

T. Douvropoulos Geometric McKay Correspondence 2/22/2016

Kepler: $\underbrace{\text{Saturn Jupiter}}_{\text{cube}}$ $\underbrace{\text{Jupiter Mars}}_{\text{tetrahedron}}$ $\underbrace{\text{Mars Earth}}_{\text{dodecahedron}}$ $\underbrace{\text{Earth Venus}}_{\text{icosahedron}}$ $\underbrace{\text{Venus Mercury}}_{\text{octahedron}}$

Algebraic McKay correspondence

$$\underset{\text{finite}}{\Gamma} \subset SL(2,\mathbb{C}) \longrightarrow \text{extended Dynkin diagram}$$

$$\text{irreps} \longrightarrow \text{vertices}$$

Geometric McKay correspondence

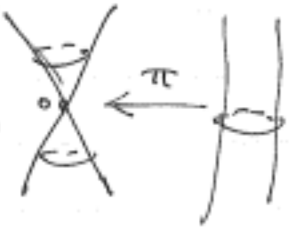
$$\underset{\text{finite}}{\Gamma} \subset SL(2,\mathbb{C}) \longrightarrow \underset{\substack{\text{unique}\\ \text{singularity}\\ \text{embedded in } \mathbb{C}^3}}{\mathbb{C}^2/\Gamma} \longrightarrow \underset{\text{Dynkin diagram}}{\text{(nonextended)}}$$

irred. components of the exceptional divisor ⟶ vertices

Two pictures:

A_1: $C_2 = \{\pm 1\} \subset SL(2,\mathbb{C})$

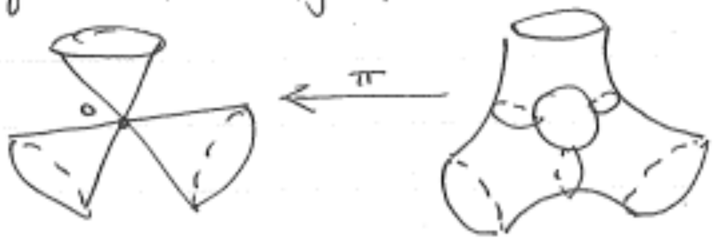
defining eqn. of $\mathbb{C}^2/\Gamma$ $x^2 - yz = 0$ (equiv. to $x^2 - y^2 - z^2 = 0$)

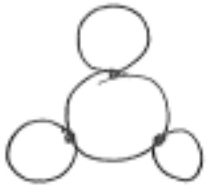


$\pi^{-1}(0) =$ 

$A_1 =$ ∘ Dynkin diagram

D_4: defining eqn. of $\mathbb{C}^2/\Gamma$ $x(y^2 - x^2) + z^2 = 0$



$E = \pi^{-1}(0) =$  $\xrightarrow[\text{diagram}]{\text{intersection}}$  D_4 Dynkin diagram

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§1 Invariant theory

Finite group G acts via lin. transformation on $V = \mathbb{C}^n$,

also acts on $\mathbb{C}[V] := \mathbb{C}[x_1, \dots, x_n]$

via $g.f = f(g^{-1}x)$

Consider invariant subalgebra $\mathbb{C}[V]^G = \{ f \in \mathbb{C}[V] : f(x) = f(g^{-1}x) \ \forall g \in G \}$

FACT (Noether-Hilbert) $\mathbb{C}[V]^G$ is generated by finitely many polynomials

FACT (Klein-DuVal)

$\underset{\text{finite}}{\Gamma} \subset SL(2,\mathbb{C}) \Rightarrow \mathbb{C}[V]^G$ is gen'd by exactly 3 polynomials

EXAMPLE 1: $\Gamma = C_N$ gen'd by $\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}$, $\zeta = e^{2\pi i/N}$

$\begin{bmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{bmatrix}(x,y) = (\zeta^{-1}x, \zeta y)$

Invariants: $\underset{f_2}{x^N}$, $\underset{f_3}{y^N}$, $\underset{f_1}{xy}$ $\qquad f_1^N - f_2 f_3 = 0$

$\underset{\text{geom. invariant theory}}{\text{G.I.T.}}$ say $\mathbb{C}[V]^\Gamma$ and V/Γ are deeply related

Indeed, $\mathbb{C}[f_1, f_2, f_3] \hookrightarrow \mathbb{C}[x,y]$ induces a map

$$\mathbb{C}^3 \longleftarrow \mathbb{C}^2$$

$$(f_1(x,y), f_2(x,y), f_3(x,y)) \longmapsfrom (x,y)$$

$$(x^N, y^N, xy)$$

(3)

This map $\mathbb{C}^2 \to \mathbb{C}^3$ realizes $\mathbb{C}^2/\Gamma$ as a topological quotient

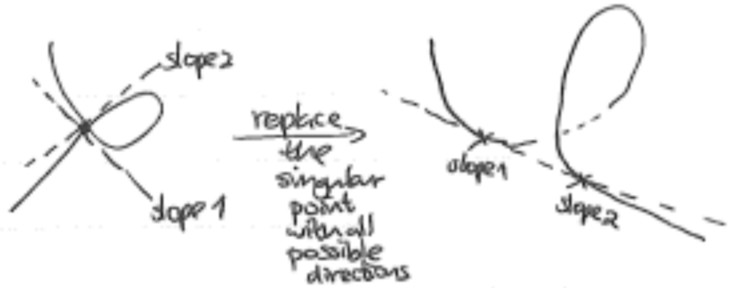
<u>EXAMPLE</u>: $\mathbb{C}^2 \longrightarrow \mathbb{C}^3 \qquad \Gamma = C_N$

$$(x,y) \longmapsto (xy, x^N, y^N)$$

Image in $\mathbb{C}^3$ with coords (x,y,z) is cut out by $x^N - yz$

§2 | Resolution of singularity

Recall



Resolving the singularity means finding $\underset{\text{smooth}}{\tilde{S}}$ and $\tilde{S} \xrightarrow{\pi} S$ such that $\tilde{S} \setminus \pi^{-1}(0) \cong S \setminus \{0\}$

Consider $B(0,\mathbb{C}^n) = \{ (v,L) \in \mathbb{C}^n \times \mathbb{P}^{n-1} \mid v \in L \}$

<u>n=3</u>: $\{ ((x,y,z), [s:t:u]) : \ xt = su, \ xu = sz, \ tz = uy \}$

$$\begin{tikzcd} & B(0,\mathbb{C}^n) \arrow[dl, "pr_1 = \beta"'] \arrow[dr, "pr_2"] & \\ \mathbb{C}^n & & \mathbb{P}^{n-1} \end{tikzcd}$$

← tells us $B(0,\mathbb{C}^n)$ is a bundle, so smooth, for example

$\beta^{-1}(\mathbb{C}^n \setminus \{0\}) \cong \mathbb{C}^n \setminus \{0\}$

"blow-up at 0"

Given $S \subset \mathbb{C}^n$ singular at 0,
then the blow-up of $0 \in S$ is $B(0,S) := \overline{\beta^{-1}(S \setminus \{0\})}$ ← Zariski closure in $\beta(0,\mathbb{C}^n)$

Exceptional <u>divisor</u> $E := B(0,S) \cap \beta^{-1}(0)$

Affine charts for $n=3$:

$$\begin{tikzcd}
(x,t,u) \arrow[r, mapsto] & ((x,xt,xu),[1:t:u]) \\
\tilde{\mathcal{U}}_1 \arrow[r, hook] & B(0,\mathbb{C}^3) \arrow[d, "\beta"] \\
(x,t,u) \arrow[dr, leftrightarrow] & \mathbb{C}^3 \\
 & (x,xt,xu)
\end{tikzcd}$$

$$\begin{tikzcd}
\tilde{\mathcal{U}}_2 \arrow[r, hook] & B(0,\mathbb{C}^3) \arrow[d] \\
(s,y,u) \arrow[dr, leftrightarrow] & \mathbb{C}^3 \\
 & (ys,y,yu)
\end{tikzcd}
\qquad
\begin{tikzcd}
\tilde{\mathcal{U}}_3 \arrow[r, hook] & B(0,\mathbb{C}^3) \arrow[d] \\
(s,t,z) \arrow[dr, leftrightarrow] & \mathbb{C}^3 \\
 & (sz,tz,z)
\end{tikzcd}$$

Resolution of type A_N, $\Gamma = C_{N+1}$

singularity has equation $x^{N+1} - yz = 0$ in $\mathbb{C}^3$,

equivalent to $x^{N+1} - y^2 + z^2 = 0$

In $\tilde{\mathcal{U}}_1$, $x^{N+1} + (xt)^2 + (xu)^2 = 0$

$\Rightarrow x^2(x^{N-1} + t^2 + u^2) = 0$

If $x=0$, get all of $((0,0,0),[1:t:u])$, which corresponds to $\beta^{-1}(0)\cap\tilde{\mathcal{U}}_1$

$x^{N-1} + t^2 + u^2 = 0$ should give me

$\overline{\beta^{-1}(S\setminus\{0\})}\cap\tilde{\mathcal{U}}_1$

$N=1 \Rightarrow 1+t^2+s^2=0$

E is given in $\tilde{\mathcal{U}}_1$ by $[1:t:s]$ such that $1+t^2+s^2=0$

(in $\tilde{\mathcal{U}}_2$ would get $s^2(ys)^{N-1}+1+u^2=0$,

and $N=1 \Rightarrow s^2+1+u^2=0$)

$\Rightarrow E_1$ is just $s^2+t^2+u^2=0$, a copy of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

$N>1 \Rightarrow E\cap\tilde{\mathcal{U}}_1 = \{ t^2+u^2=0 \} = \{ [1:a:\pm ia] : a\in\mathbb{C} \}$

$E\cap\tilde{\mathcal{U}}_2 = \{ s^{N+1}y^{N-1}+u^2=0 \text{ at } y=0 \} = \{ [s:1:u] : 1+u^2=0 \}$

$= \{ [b:1:\pm i] \}$

So $E = \{[1:a:\pm ia]\}$ and $\{[b:1:\pm i]\}$

If $b = \frac{1}{a}$, $[1:a:\pm ia] = [b:1:\pm i]$

E has 2 lines, 

but we have not yet resolved the singularity if $n \geq 2$

since the equation $x^{N+1} + t^2 + u^2 = 0$ is still singular.

You proceed inductively, and keep blowing up to get

⟷ type A_N Dynkin diagram

A schematic picture: (Reference:
Givental "Reflection groups in Singularity Theory"
Trans. Amer. Math. Soc.
153, 1992
– hard to access!")