ALGEBRAIC COMBINATORICS:

Using algebra to help one count

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COMBINATORICS
= study of finite or discrete objects
and their structure,
including counting them
(= ENUMERATIVE COMBINATORICS)

Part of
ALGEBRAIC COMBINATORICS

is using algebra to help you do
ENUMERATIVE COMBINATORICS

EXAMPLE:
Enumerating subsets of a set,
up to symmetry.

We'll observe some interesting properties,
some easy,
some harder.
Consider the finite set
\[ [n] := \{1, 2, \ldots, n\} \]
with some set $G$ of symmetries
\[ (= \text{a subgroup of the symmetric group } S_n \text{ on } n \text{ letters}) \]

e.g. \( n = 6 \)

\[ G = \text{rotational symmetries} = \text{cyclic group } C_6 \]

\[ = \{ \text{rotations through} \]
\[ 0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ \} \]
Let's count

\[ 2^n : = \text{subsets of } [n] \]

\[ \uparrow \]

black-white colorings of \([n]\)

up to equivalence by elements of \(G\),

i.e. \(G\)-orbits of subsets of \([n]\)

\[ =: 2^n / G \]

e.g. for \(G = C_6\) as above,

\(G\)-orbits in \(2^n / G\) are sometimes called black-white necklaces:
let's even be more refined...

\[ (\binom{n}{k}) = k\text{-element subsets of } [n] \]

The symmetries \( G \) also permute \( (\binom{n}{k}) \).

Can we count:

\[ (\binom{n}{k})_G := G\text{-orbits of } k\text{-subsets of } [n] \]

Let \( \text{c}_k := | (\binom{n}{k})_G | \)

Q: What can we say in general about \( \text{c}_0, \text{c}_1, \ldots, \text{c}_n \)?

A: They share a number of properties with binomial coefficients \( \binom{0}{k}, \binom{1}{k}, \ldots, \binom{n}{k} \) ...

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**EXAMPLE:** \( G = C_6 \)

| \( k \) (= # of blacks) | \( C_6 \)-orbits on \( (U_{61}) \) | \( C_k = \left| \frac{(U_{61})}{C_6} \right| \) |
|--------------------------|---------------------------------|-----------------|
| 6                        | ![Circle with BB and BBB]        | 1               |
| 5                        | ![Circle with WBB and BBB]      | 1               |
| 4                        | ![Circles with WBWB, BBWB, and WWBB] | 3               |
| 3                        | ![Circles with WWB, WBB, BWW, and BBW] | 4               |
| 2                        | ![Circles with WBB and WWB]     | 3               |
| 1                        | ![Circle with WWWW]             | 1               |
| 0                        | ![Circle with WWWW]             | 1               |
6 overlays

#1

1

#2

1

SYMMETRY

1

3

1

4

3

1

vi

vi

vi

vi

Unimodality

#3

+1

-1

+3

-4

+3

-1

+1

+2

Alternating sum

(\text{black} \leftrightarrow \text{white})

= \# \text{ of self-complementary } G\text{-orbits}

\text{same necklace}

\text{swap black \& white}

\text{swap black \& white}

\text{same necklace}
The generating function

\[ \sum_{k=0}^{n} c_k q^k = c_0 + c_1 q + c_2 q^2 + \ldots + c_n q^n \]

is simply the average over the group \( G \)
of some easy products (recording cycle sizes)

\[ G = \{ \begin{cases} \circ \circ \circ & (1+q)^6 \\ \circ \circ \circ \circ & (1+q)^1 \\ \circ \circ \circ \circ \circ & (1+q)^2 \\ \circ \circ \circ \circ \circ \circ & (1+q^3)^2 \\ \circ \circ \circ \circ \circ \circ \circ & (1+q^6)^1 \end{cases} \]
THEOREM:

For any subgroup $G$ of $S_n$ and $c_k = \binom{[n]}{k}/G$, the sequence $c_0, c_1, \ldots, c_n$

(1) has GENERATING FUNCTION

$$\sum_{k=0}^{n} c_k q^k = \frac{1}{|G|} \sum_{g \in G} \prod_{c \in G} (1 + q^{c_g})$$

(Ryser, Rentfrow)

(2) has ALTERNATING SUM

$$c_0 - c_1 + c_2 - \ldots + (-1)^n c_n = \# \text{ of self-complementary } G \text{-orbits}$$

(de Bruijn)

(3) is SYMMETRIC: $c_i = c_{n-i}$

(nearly obvious)

(4) is UNIMODAL: $c_0 \leq c_1 \leq c_2 \leq \ldots \leq c_{\lfloor \frac{n}{2} \rfloor}$

(Stanley)

(1) - (3) are not hard to show directly

(4) is hard (in fact, not known) without using some sort of linear algebra, algebra or representation theory, but easy with them.
**Unified Proof Idea for (1) – (4)**

Re-interpret subsets or black/white colorings and G-orbits in terms of (multi-)linear algebra...

Let $V = \mathbb{C}^2$ with C-basis $\{ \mathbf{w}, \mathbf{b} \}$ “white” “black”

Then $V \otimes^n = V \otimes V \otimes \cdots \otimes V$

$n$ times

has C-basis elements $e_S$ indexed by subsets $S \subseteq [n]$

E.g. $n=6$

$S = \{1,3,5\}$

$e_S = \mathbf{b} \mathbf{w} \mathbf{b} \mathbf{w} \mathbf{b} \mathbf{w}$, or just $\mathbf{b} \mathbf{w} \mathbf{b} \mathbf{w} \mathbf{b} \mathbf{w}$
$G$ acts on $V^\otimes n$ by permuting the tensor positions.

$$(V^\otimes n)^G := \text{the } G\text{-fixed subspace of } V^\otimes n$$

has $C$-basis indexed by $G$-orbits of subsets or black-white colorings

e.g. $n=6$

$G = C_6$

\[
\begin{array}{cccc}
  B & B & w & w \\
  B & W & w & B
\end{array}
\]

\[\leftrightarrow \quad wbwbw+bwbwwb \in (V^\otimes 6)^G\]
\( V^\otimes n = \bigoplus_{k=0}^{n} (V^\otimes n)_k \) is a graded \( \mathbb{C} \) vector space, span of \( e_S \) with \( |S|=k \)

and \( G \) acts on each graded component \((V^\otimes n)_k\)

so similarly

\[
(V^\otimes n)_G = \bigoplus_{k=0}^{n} (V^\otimes n)_k^G .
\]

with basis corresponding to \( (\binom{\mu}{k})^G \)

Thus \( c_k = \dim_{\mathbb{C}} (V^\otimes n)_k^G \)

which gives a good starting point...
Proof of (3) (Symmetry): \( C_k = C_{n-k} \)

(A bit silly, but contains an idea useful for proof of (2)!)

Consider \( \sigma := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in GL_2(\mathbb{C}) = GL(V) \)

acting on \( V = \mathbb{C}^2 \) by swapping black \( b \) \( \leftrightarrow \) white \( w \)

\( GL(V) \) also acts \( \text{diagonally} \) on \( V^\otimes n \):

\[
\begin{align*}
&v_1 \otimes \cdots \otimes v_n \mapsto \sigma(v_1) \otimes \cdots \otimes \sigma(v_n) \\
&b b b w w b \mapsto w w w b b b
\end{align*}
\]

and this commutes with the \( G \)-action,

so \( \sigma \) preserves \( (V^\otimes n)_G \).

In fact, it gives a \( \mathbb{C} \)-vector space isomorphism

\[
(V^\otimes n)_G \cong (V^\otimes n)_{n-k}
\]

so \( \dim(\mathbb{C}, (V^\otimes n)_G)_k = \dim(\mathbb{C}, (V^\otimes n)_G)_{n-k} \)

\( \text{i.e.} \)

\[
C_k = C_{n-k}
\]
Sketch proof of (2) (alternating sum):

\[ c_0 - c_1 + c_2 - \ldots + (-1)^n c_n = \# \text{ self-complementary } G\text{-orbits in } 2^{[n]} \]

Note that \( \sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) is diagonalizable with eigenvalues +1, -1, so it is conjugate in GL(V) to \( \begin{bmatrix} w & b \\ 0 & -1 \end{bmatrix} \)

Hence \( \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix} \) should act with the same trace on \( (V \otimes \delta)^G \).

Note that \( \sigma = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) permutes the basis elements for \( (V \otimes \delta)^G \) that are indexed by \( G\)-orbits of subsets, via complementation

\[ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \]

Hence

\[ \text{Trace}(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) : (V \otimes \delta)^G \rightarrow (V \otimes \delta)^G \]

\[ = \# \text{ self-complementary } G\text{-orbits in } 2^{[n]} \]
On the other hand,

\[
\begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} ^{b} \cdot \begin{pmatrix}
\omega \\
b
\end{pmatrix} \quad \omega \mapsto \omega, \quad b \mapsto -b
\]

so \( \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \) acts on \((\bigwedge \omega)^k\) by the scalar \((-1)^k\)

E.g. \( \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} : b b b w w b \mapsto (b)(-b)(-b)w(w)(-b) \)
\[= (-1)^k b b b w w b\]

Hence \( \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} \) acts on \((\bigwedge \omega)^G\) by the scalar \((-1)^k\)

and since \((\bigwedge \omega)^G = \bigoplus_{k=0}^{n} (\bigwedge \omega)^G\)

Trace \( \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix} : (\bigwedge \omega)^G \to (\bigwedge \omega)^G \) =

\[+ \dim C(\bigwedge \omega)^G_0 - \dim C(\bigwedge \omega)^G_1 + \dim C(\bigwedge \omega)^G_2 - \ldots \]

\[= c_0 \quad c_1 \quad + \quad c_2 \quad - \ldots\]
Sketch proof of (4) (UNIMODALITY): \( c_0 \leq c_1 \leq \ldots \leq c_{\lfloor n/2 \rfloor} \)

We want \( c_k \leq c_{k+1} \) for \( k < \frac{n}{2} \)

i.e. \( \dim_c (V^\otimes n)^G_k \leq \dim_c (V^\otimes n)^G_{k+1} \)

so maybe there's a \( C \) vector space injection

\[
(V^\otimes n)^G_k \hookrightarrow (V^\otimes n)^G_{k+1}
\]

for \( k < \frac{n}{2} \)

In fact, maybe it comes from a \( G \)-equivariant injection

\[
(V^\otimes n)_k \xrightarrow{U_k} (V^\otimes n)_{k+1}
\]

for \( k < \frac{n}{2} \)

Here's an obvious guess for defining \( U_k \):

\[
e_g \xrightarrow{e_g} \sum_{T: T \supset S, \quad |T| = k+1} e_T
\]

for \( |S| = k \)

e.g. \( bwbwbbw \xrightarrow{U_k} bwbwbbw + bbwbwb + bbwbwb \)

It's easily seen to be \( G \)-equivariant,

but why is it injective for \( k < \frac{n}{2} \) ?
Here is one (of several) linear algebra arguments for the injectivity of $U_k$ if $k < \frac{n}{2}$:

Check that

$$U_k^* U_k - U_{k-1}^* U_{k-1} = (n-2k)I_{(V^{\otimes n})_k}$$

for $k = 1, 2, \ldots, n$.

So

$$U_k^* U_k = U_{k-1}^* U_{k-1} + (n-2k)I_{(V^{\otimes n})_k}$$

positive semidefinite

positive definite because $k < \frac{n}{2}$

$$\Downarrow$$

$U_k^* U_k$ positive definite, hence invertible

$$\Downarrow$$

$U_k$ injective.
Sketch proof of (1) (GENERATING FUNCTION):
\[
\sum_{k=0}^{m} C_k q^k = \frac{1}{|G|} \sum_{g \in G} \prod_{c \in C} (1 + q |C|)
\]

When a finite group $G$ acts linearly on a $C$-vector space $U$, the averaging operator
\[
\tau_G = \frac{1}{|G|} \sum_{g \in G} \tau_g
\]
is an idempotent projector $\tau_G : U \rightarrow U^G$
\[
(\tau_G^2 = \tau_G)
\]
Hence its trace computes the dimension of the image:
\[
\text{Trace} (\tau_G : U \rightarrow U) = \dim_U (U^G).
\]
Let's apply this with $U = (V^\otimes n)_k$

to compute $\dim_U (U^G) = \dim_U (V^\otimes n)_k = c_k.$
\[
C_\omega = \dim_\mathbb{C}(V^{\otimes n})_\omega^G \\
= \text{Trace} \left( \frac{1}{|G|} \sum_{g \in G} g : (V^{\otimes n})_\omega^G \rightarrow (V^{\otimes n})_\omega^G \right) \\
= \frac{1}{|G|} \sum_{g \in G} \text{Trace} \left( g : (V^{\otimes n})_\omega^G \rightarrow (V^{\otimes n})_\omega^G \right)
\]

Note: \( g \) permutes the basis elements \( e_\omega \) for \( (V^{\otimes n})_\omega^G \).

\[e.g. \ g = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}
\]
sends \( \begin{array}{cccccc}
\omega & \omega & \omega & \omega & \omega & \omega
\end{array} \rightarrow \begin{array}{cccccc}
b & w & w & b & w & w
\end{array} \)

and so its trace counts black/white colorings of \([n]\) that it fixes, i.e. those which are monochromatic on each of \( g \)'s cycles.

\[e.g. \ \begin{array}{cccccc}
\omega & \omega & \omega & \omega & \omega & \omega
\end{array} \rightarrow \begin{array}{cccccc}
\omega & \omega & \omega & \omega & \omega & \omega
\end{array} \]

\[1 + 2q^3 + 1q^6 = (1 + q^3)^2\]

Hence
\[
C_\omega = \frac{1}{|G|} \sum_{g \in G} \left( \text{\# of black/white colorings of } [n] \right) \\
\text{with } k \text{ blacks, monochromatic on } g \text{'s cycles}
\]

\[= \frac{1}{|G|} \sum_{g \in G} \left[ \text{coefficient of } q^k \text{ in } (1+q|C_1|) \right] \text{ of } g\]

i.e.
\[
\sum_{k=0}^{n} C_\omega q^k = \frac{1}{|G|} \sum_{g \in G} \prod_{\text{cycles } C} (1+q|C_1|).
\]