# A generalization of the Chevalley-Mitchell Theorem

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# Outline

I. Chevalley, Mitchell and the the theorem we could prove.

II. Springer and the conjecture we really want to prove

III. (Why might one care?)

## I. Chevalley, Mitchell

Let $\mathbb{F}$ be an arbitrary field,
and $V$ an $n$-dimensional vector space over $\mathbb{F}$.

We'll define a reflection group
to be a finite subgroup

$$W \subset GL(V) \cong GL_n(\mathbb{F})$$

for which the $W$-action on
the symmetric algebra

$$S := \mathrm{Sym}(V^*) \cong \mathbb{F}[x_1, \ldots, x_n]$$

has invariant subring $S^W$ a polynomial algebra:

$$S^W = \mathbb{F}[f_1, \ldots, f_n].$$

Theorem(Serre 1967)
For finite subgroups $W \subset GL_n(\mathbb{F})$,
$S^W$ polynomial implies
$W$ is generated by reflections.

Here reflection means the codimension of its fixed space is 1.

Examples:
$W = \mathfrak{S}_n =$ symmetric groups
$W = G(d, e, n)=$ monomial groups
$W = GL_n(\mathbb{F}_q) =$ finite general linear groups

For $|W| \in \mathbb{F}^{\times}$, Chevalley (and Shephard-Todd) had shown the converse also holds, plus an interesting feature of the coinvariant algebra

$$S/(S^W_+) = S/(f_1, \ldots, f_n).$$

Theorem (Chevalley 1955)
For $S^W$ polynomial and $|W| \in \mathbb{F}^{\times}$,
one has an isomorphism of $W$-representations

$$S/(S^W_+) \underset{\mathbb{F}[W]-mod}{\cong} \mathbb{F}[W]$$

Proof idea:
Apply the Normal Basis Theorem
to the Galois extension

$$\text{Frac}(S^W) = \text{Frac}(S)^W \hookrightarrow \text{Frac}(S).$$

So what? There are few groups $W$ with $S^W$ polynomial, after all...

Well, when $|W| \in \mathbb{F}^\times$, the isomorphism

$$S/(S^W_+) \underset{\mathbb{F}[W]-mod}{\cong} \mathbb{F}[W].$$

gives a consequence for any subgroup $W' \subset W$: restrict to the $W'$-fixed subspaces...

$$\begin{array}{ccc} (S/(S^W_+))^{W'} & \underset{\mathbb{F}[N_W(W')]-mod}{\cong} & \mathbb{F}[W]^{W'} \\ \| & & \| \\ S^{W'}/(S^W_+) & & \mathbb{F}[W/W'] \end{array}$$

where $N_W(W')$ is the normalizer of $W'$ in $W$.

Sounds good, but what about
when $|W| \notin \mathbb{F}^{\times}$, like $W = GL_n(\mathbb{F}_q)$?

Theorem(Mitchell 1985)
For $S^W$ polynomial one has a Brauer-isomorphism of $W$-representations

$$S/(S_{+}^{W}) \underset{\mathbb{F}[W]-mod}{\approx} \mathbb{F}[W]$$

In other words, they have
the same composition factors.

Unfortunately, given only a Brauer-isomorphism, you can't equate $W'$-fixed spaces.

Also, since $W' \subset W$ might have $S^{W'}$ not Cohen-Macaulay, one shouldn't look only at

$$S^{W'}/(S^W_+) = S^{W'} \otimes_{S^W} \mathbb{F}$$
$$= \mathrm{Tor}_0^{S^W}(S^{W'}, \mathbb{F}).$$

where $\mathbb{F} := S^W/(S^W_+)$ as $S^W$-module.

One should look at the rest of $\mathrm{Tor}_i$ !

Theorem(−, Smith, Webb)
For $S^W$ polynomial and any subgroup $W' \subset W$, one has a virtual-Brauer-isomorphism of $N_W(W')$-representations

$$\mathrm{Tor}_*^{S^W}(S^{W'},\mathbb{F}) \underset{\mathbb{F}[N_W(W')]-mod}{\approx} \mathbb{F}[W/W']$$

where virtual means the left side is

$$\sum_{i \geq 0} (-1)^i \mathrm{Tor}_i^{S^W}(S^{W'},\mathbb{F})$$

in an appropriate Grothendieck group.

Proof idea
Re-work homologically Chevalley's proof via Normal Basis Theorem.

II. Springer and the conjecture we want.

When $S^W$ is polynomial, call $c \in W$ a regular element if it has an eigenvector

$$v \in \overline{V} := V \otimes_{\mathbb{F}} \overline{\mathbb{F}}$$

fixed by no reflections.

Call the eigenvalue $\omega \in \overline{\mathbb{F}}^{\times}$ for $c$ when acting on $v$ its regular eigenvalue.

Theorem(Springer 1972)
Assume $S^W$ polynomial and $|W| \in \mathbb{F}^\times$.
Let $C = \langle c \rangle$ for some regular element $c$,
with regular eigenvalue $\omega^{-1}$.
Then one has an isomorphism of
$W \times C$-representations

$$S/(S^W_+) \underset{\overline{\mathbb{F}}[W \times C]-mod}{\cong} \overline{\mathbb{F}}[W].$$

Here on $S/(S^W_+)$,

- $W$ acts by linear substitutions,
- $C$ acts by scalar substitutions

$$c(x_i) = \omega x_i$$
$$c(f) = \omega^d f \text{ if } \deg(f) = d,$$

while on $\overline{\mathbb{F}}[W]$ the groups $W, C$ act by
left, right-multiplication.

When $|W| \in \mathbb{F}^{\times}$, one can again take $W'$-fixed spaces in Springer's theorem, giving an isomorphism of $N_W(W') \times C$-representations

$$S^{W'}/(S^W_+) \underset{\overline{\mathbb{F}}[N_W(W') \times C]-mod}{\cong} \overline{\mathbb{F}}[W/W']$$

Just equating the $C$-character on both sides already gives a great combinatorial consequence...

(III. Why might we care?)

Theorem 1:
(−,Stanton,White 2004, −,Stanton,Webb 2005)
Let $W \subset GL(V)$ be finite with $S^W$ polynomial, and $C$ the cyclic subgroup generated by a regular element. **Assuming** $|W| \in \mathbb{F}^\times$, the triple $(X, X(q), C)$

$X =$ any set with transitive $W$-action, say $X = W/W'$

$$X(q) = \frac{\text{Hilb}(S^{W'}, q)}{\text{Hilb}(S^W, q)}$$

$C$ translating the cosets $wW'$ in $X$

exhibits the cyclic sieving phenomenon: for any $c \in C$ and $\omega$ a root-of-unity one has

$$|X^c| = [X(q)]_{q=\omega}\,.$$

We suspect one does not need $|W| \in \mathbb{F}^\times$.
This would follow from...

Conjecture: When $S^W$ is polynomial, for any subgroup $W' \subset W$, one has a virtual Brauer-isomorphism of $N_W(W') \times C$-representations

$$\mathrm{Tor}_*^{S^W}(S^{W'}, \overline{\mathbb{F}}) \underset{\overline{\mathbb{F}}[N_W(W') \times C]-mod}{\approx} \overline{\mathbb{F}}[W/W'].$$

Known:

– without $C$-action (the earlier theorem).
– for $W' = 1$ (–,Stanton,Webb, 2005)

Whence the combinatorial consequence?

$$\color{red}{X(q)} := \frac{\text{Hilb}(S^{W'}, q)}{\text{Hilb}(S^{W}, q)}$$
$$= \sum_{i \geq 0} (-1)^i \text{Hilb}(\text{Tor}_i^{S^W}(S^{W'}, \overline{\mathbb{F}}), q)$$

Why might you, the invariant theorist, care?

For $W' \subset W = GL_n(\mathbb{F}_q)$ it's tough to compute $\text{Hilb}(S^{W'}, t)$. But believing the the conjecture, an easy, fast `GAP` computation using regular elements in $GL_n(\mathbb{F})$ gives

$$\frac{\text{Hilb}(S^{W'}, t)}{\text{Hilb}(S^{W}, t)} \quad \text{mod } t^{q^n-1} - 1.$$