A generalization of the Chevalley-Mitchell Theorem

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Outline

I. Chevalley, Mitchell and the theorem we could prove.

II. Springer and the conjecture we really want to prove

III. (Why might one care?)
Let $\mathbb{F}$ be an arbitrary field, and $V$ an $n$-dimensional vector space over $\mathbb{F}$.

We’ll define a **reflection group** to be a finite subgroup

$$W \subset GL(V) \cong GL_n(\mathbb{F})$$

for which the $W$-action on the symmetric algebra

$$S := \text{Sym}(V^*) \cong \mathbb{F}[x_1, \ldots, x_n]$$

has invariant subring $S^W$ a polynomial algebra:

$$S^W = \mathbb{F}[f_1, \ldots, f_n].$$
Theorem (Serre 1967)
For finite subgroups $W \subset GL_n(\mathbb{F})$, $S^W$ polynomial implies $W$ is generated by reflections.

Here reflection means the codimension of its fixed space is 1.

Examples:
$W = \mathcal{G}_n = \text{symmetric groups}$
$W = G(d, e, n) = \text{monomial groups}$
$W = GL_n(\mathbb{F}_q) = \text{finite general linear groups}$
For \(|W| \in \mathbb{F}^\times\), Chevalley (and Shephard-Todd) had shown the converse also holds, plus an interesting feature of the coinvariant algebra

\[
S/(S^W_+) = S/(f_1, \ldots, f_n).
\]

**Theorem** (Chevalley 1955)
For \(S^W\) polynomial and \(|W| \in \mathbb{F}^\times\), one has an isomorphism of \(W\)-representations

\[
S/(S^W_+) \cong \mathbb{F}[W]_{\mathbb{F}[W] \text{-mod}}.
\]

**Proof idea:**
Apply the Normal Basis Theorem to the Galois extension

\[
\text{Frac}(S^W) = \text{Frac}(S)^W \hookrightarrow \text{Frac}(S).
\]
So what? There are few groups $W$ with $S^W$ polynomial, after all...

Well, when $|W| \in \mathbb{F}^\times$, the isomorphism

$$S/(S^W) \cong_{\mathbb{F}[W]-mod} F[W].$$

gives a consequence for any subgroup $W' \subset W$: restrict to the $W'$-fixed subspaces...

$$\frac{(S/(S^W))^{W'}}{\mathbb{F}[N_W(W')]-mod} \cong_{\mathbb{F}[W]^{W'}} \frac{S^{W'}/(S^W)}{\mathbb{F}[W/W']}$$

where $N_W(W')$ is the normalizer of $W'$ in $W$. 
Sounds good, but what about when \(|W| \not\in \mathbb{F}^\times\), like \(W = GL_n(\mathbb{F}_q)\)?

**Theorem** (Mitchell 1985)
For \(S^W\) polynomial one has a **Brauer**-isomorphism of \(W\)-representations

\[
\frac{S}{(S^W)} \cong \frac{\mathbb{F}[W]}{\mathbb{F}[W]} \mod \mathbb{F}[W]
\]

In other words, they have the same composition factors.
Unfortunately, given only a Brauer-isomorphism, you can’t equate $W'$-fixed spaces.

Also, since $W' \subset W$ might have $S^{W'}$ not Cohen-Macaulay, one shouldn’t look only at

$$S^{W'}/(S^W_+) = S^{W'} \otimes_{S^W} F = \text{Tor}^S_0 (S^{W'}, F).$$

where $F := S^W/(S^W_+)$ as $S^W$-module.

One should look at the rest of $\text{Tor}_i$!
**Theorem** (–, Smith, Webb)
For $S^W$ polynomial and any subgroup $W' \subset W$, one has a virtual-Brauer-isomorphism of $N_W(W')$-representations

$$\text{Tor}_*^{S^W}(S^{W'}, \mathbb{F}) \cong \mathbb{F}[N_W(W')]-\text{mod} \mathbb{F}[W/W']$$

where virtual means the left side is

$$\sum_{i \geq 0} (-1)^i \text{Tor}_i^{S^W}(S^{W'}, \mathbb{F})$$

in an appropriate Grothendieck group.

**Proof idea**
Re-work homologically Chevalley’s proof via Normal Basis Theorem.
II. Springer and the conjecture we want.

When $S^W$ is polynomial, call $c \in W$ a regular element if it has an eigenvector

$$v \in \overline{V} := V \otimes_{F} \overline{F}$$

fixed by no reflections.

Call the eigenvalue $\omega \in \overline{F}^\times$ for $c$ when acting on $v$ its regular eigenvalue.
**Theorem** (Springer 1972)
Assume $S^W$ polynomial and $|W| \in \mathbb{F}^\times$.
Let $C = \langle c \rangle$ for some regular element $c$, with regular eigenvalue $\omega^{-1}$.
Then one has an isomorphism of $W \times C$-representations

$$S/(S^W_+) \cong \overline{\mathbb{F}[W \times C]} \mod \mathbb{F}[W].$$

Here on $S/(S^W_+)$,
- $W$ acts by linear substitutions,
- $C$ acts by scalar substitutions

$$c(x_i) = \omega x_i$$

$$c(f) = \omega^d f \text{ if } \deg(f) = d,$$

while on $\overline{\mathbb{F}[W]}$ the groups $W, C$ act by left, right-multiplication.
When $|W| \in \mathbb{F}^\times$, one can again take $W'$-fixed spaces in Springer’s theorem, giving an isomorphism of 
$N_W(W') \times C$-representations

$$S^{W'}/(S^W) \cong \frac{\overline{\mathbb{F}}[N_W(W') \times C]}{\overline{\mathbb{F}}[W/W']}$$

Just equating the $C$-character on both sides already gives a great combinatorial consequence...
(III. Why might we care?)

**Theorem 1:**


Let $W \subset GL(V)$ be finite with $S^W$ polynomial, and $C$ the cyclic subgroup generated by a regular element. **Assuming** $|W| \in \mathbb{F}^\times$, the triple $(X, X(q), C)$

\[
X = \text{any set with transitive } W\text{-action, say } X = W/W'
\]

\[
X(q) = \frac{\text{Hilb}(S^{W'}, q)}{\text{Hilb}(S^W, q)}
\]

$C$ translating the cosets $wW'$ in $X$

exhibits the **cyclic sieving phenomenon**: for any $c \in C$ and $\omega$ a root-of-unity one has

\[
|X^c| = [X(q)]_{q=\omega}.
\]
We suspect one does not need $|W| \in \mathbb{F}^\times$. This would follow from...

**Conjecture:** When $S^W$ is polynomial, for any subgroup $W' \subset W$, one has a virtual Brauer-isomorphism of $N_W(W') \times C$-representations

$$\text{Tor}^*_S(S^{W'}, \overline{\mathbb{F}}) \approx \overline{\mathbb{F}}[N_W(W') \times C]_{-\text{mod}}.$$

**Known:**
- without $C$-action (the earlier theorem).
- for $W' = 1$ (–, Stanton, Webb, 2005)
Whence the combinatorial consequence?

\[
X(q) := \frac{\text{Hilb}(S^W, q)}{\text{Hilb}(S^W, q)}
= \sum_{i \geq 0} (-1)^i \text{Hilb}(\text{Tor}_i^{S^W} (S^{W'}, \overline{F}), q)
\]

Why might you, the invariant theorist, care?

For \( W' \subset W = GL_n(\mathbb{F}_q) \) it's tough to compute \( \text{Hilb}(S^W, t) \). But believing the conjecture, an easy, fast GAP computation using regular elements in \( GL_n(\mathbb{F}) \) gives

\[
\frac{\text{Hilb}(S^{W'}, t)}{\text{Hilb}(S^W, t)} \pmod{t^{q^n-1} - 1}.
\]