Cyclic symmetry

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Smith College
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Everyone enjoys symmetry!
3-fold

5-fold

I like it cyclic.
Better yet, I like counting formulas for objects with cyclic symmetry, particularly when they come from formulas that were already there ...
Let's count $k$-element subsets of $\{1,2,\ldots,n\}$

**Example:**  \[ k = 2 \quad n = 4 \]

There are \[ \binom{n}{k} = \binom{4}{2} = 6 \]

of them:

\[ \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\} \]
\[ \binom{n}{k} = \text{the binomial coefficient} \]

\[ = \frac{n!}{k!(n-k)!} \]

where \( n! = n(n-1)\cdots3 \cdot 2 \cdot 1 \)

e.g. \( \binom{4}{2} = \frac{4!}{2!2!} = \frac{4 \cdot 3 \cdot 2 \cdot 1}{2 \cdot 1 \cdot 2 \cdot 1} = 6 \checkmark \)

Pascal:

\[ \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \]
Pascal's triangle

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
\end{array}
\]

\[
\binom{3}{1} + \binom{3}{2} = \binom{4}{2}
\]
Fine, but can subsets have cyclic symmetry?

Yes, if we place \( \{1, 2, ..., n\} \) on a circle.....

\[ \begin{array}{cccccccc}
11 & 12 & 1 & 10 & 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2
\end{array} \]
Every 6-element subset of \([1,2,\ldots,12]\) has 1-fold symmetry.

E.g. \([1/2,4,7,8,9]\)

11 12 1
10
9
8
7 6 5

Some have 3-fold symmetry:

11 12 1
10
9
8
7 6 5

Some even have 6-fold symmetry:

11 12 1
10
9
8
7 6 5

E.g. \([1,3,4,7,8,10]\)
How many $k$-element subsets of $\{1,2,\ldots,n\}$ have $d$-fold symmetry?

**THEOREM** (Stanton, White, R. 2004)

It's what you get from the $q$-binomial coefficient

$$\binom{n}{k}_q$$

when you plug in for $q$ a primitive complex $d$th root of unity!
\[
\begin{align*}
\binom{n}{k}_q & \overset{\text{DEF.}}{=} \frac{[n]!_q}{[k]!_q [n-k]!_q} \\
\text{where } [n]!_q & \overset{\text{DEF}}{=} [n]_q [n-1]_q \cdots [3]_q [2]_q [1]_q \\
[n]_q & \overset{\text{DEF}}{=} 1 + q + q^2 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}
\end{align*}
\]

**Example:** \( k=2, \ n=4 \)

\[
\begin{align*}
\binom{4}{2}_q &= \frac{[4]!_q}{[2]!_q [2]!_q} = \frac{[4]_q [3]_q [2]_q [1]_q}{[2]_q [1]_q [2]_q [1]_q} \\
&= \frac{(1+q+q^2+q^3)(1+q+q^2)(1+q)(1)}{(1+q)(q) \cdot (1+q)(1)} \\
&= (1+q^2)(1+q+q^2) = 1+q+2q^2+q^3+q^4
\end{align*}
\]
Remember complex roots of unity?

The 4th roots of unity:
-1, i, -i, 1

±i are the primitive 4th roots of unity.

-1 is a primitive 2nd root of unity.

The 5th roots of unity:
2πi/5, 4πi/5, 6πi/5, 8πi/5, 10πi/5

-1, i, -i, 1
So we plug in $q = +1, -1, \pm i$ in the $q$-binomial coefficient

$$[\begin{array}{c} 4 \\ 2 \end{array}]_q = 1 + q + 2q + q + q$$

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</table>

7+1+2+1+1=6

1-1+2-1+1=2

1+i-2-i+1=0

2-fold symmetric

4-fold symmetric (none!)
The theory of $q$-analogues...

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1} \quad q=1 \quad \mapsto n$$

$$[n]_q! = [n]_q \cdots [2]_q [1]_q \quad q=1 \quad \mapsto n!$$

$$[n]_q^k = \frac{[n]_q!}{[k]_q! [n-k]_q!} \quad q=1 \quad \mapsto \binom{n}{k}$$

is a very well-studied, well-developed and fascinating subject.
The $q$-binomials $\left[ \begin{array}{c} n \\ k \end{array} \right]_q$ have beautiful properties and many interpretations, e.g.,

- $\left[ \begin{array}{c} n \\ k \end{array} \right]_q$ is a polynomial in $q$, with nonnegative coefficients
  
  e.g. $\left[ \begin{array}{c} 4 \\ 2 \end{array} \right]_q = 1 + q + 2q^2 + q^3 + q^4$

- $\left[ \begin{array}{c} n \\ k \end{array} \right]_q = \left[ \begin{array}{c} n-1 \\ k \end{array} \right]_q + q^{n-k} \left[ \begin{array}{c} n-1 \\ k-1 \end{array} \right]_q$ (q-Pascal!)
  
  e.g. $\left[ \begin{array}{c} 4 \\ 2 \end{array} \right]_q = \left[ \begin{array}{c} 3 \\ 2 \end{array} \right]_q + q^2 \left[ \begin{array}{c} 3 \\ 1 \end{array} \right]_q$
  
  $= 1 + q + q^2 + \frac{q^2}{6}(1 + q + q^2)$
  
  $= 1 + q + 2q^2 + q^3 + q^4$
• $\begin{bmatrix} n \\ k \end{bmatrix}_q$ has interpretations in geometry, topology, representation theory

• When you plug in $q=p^m$ a prime power there is a finite field $\mathbb{F}_q$ with $q$ elements, and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ counts (!) the set of $k$-dimensional $\mathbb{F}_q$-linear subspaces inside $\mathbb{F}_q^n$
EXAMPLE: \( q=3 = p^1 \)

\[ \mathbb{F}_q = \mathbb{F}_3 = \mathbb{Z}/3\mathbb{Z} \]

\[ = \{ \text{integers modulo 3} \} \]

e.g. \( \bar{1} + \bar{1} + \bar{1} = \bar{0} = \bar{1} + \bar{2} \)
\[ \bar{2} \cdot \bar{2} = \bar{4} = \bar{1} \quad \text{in } \mathbb{F}_3 \]

So taking \( k=1 \) and \( n=2 \),

\[ \left[ \begin{array}{c} 2 \\ 1 \end{array} \right]_q = \frac{[2]}{[1]} \cdot [1]_q = \frac{(1+q)(1)}{(1)(1)} = 1+q \]

\[ \Rightarrow 1+3 = 4 \]

plug in \( q=3 \)
Since \([ \frac{2}{1}, m \rightarrow 4, q=3 \)\], there should be exactly 4 1-dimensional \( \mathbb{F}_3 \)-linear subspaces ("lines through the origin") in the 2-dimensional space \( \mathbb{F}_3^2 \).
\begin{align*}
slope \overline{0} &= \text{\textit{x-axis}} \\
slope \overline{1} &= \text{\textit{diagonal}} \\
slope \overline{2} &= \{(0,0), (1,2), (2,1)\} \\
slope \infty &= \text{\textit{y-axis}}
\end{align*}
The fact that \([k]_q\) can count cyclically symmetric \(k\)-element subsets of \([1,2,\ldots,n]\) has many proofs.

- Some use the connections to representation theory.
- Others are more direct, but perhaps less illuminating.

We have many, many examples where a polynomial in \(q\) counts cyclically symmetric objects when we plug in a root-of-unity for \(q\).

(The "cyclic sieving phenomenon")

Here is one that I like very much, but feel we understand poorly...
An old counting problem:
How many ways to \textbf{triangulate} (cut into triangles) a convex \textit{n}-sided polygon?

\begin{align*}
n = 3 & \quad \triangle & \quad 1 \text{ way}. \\

n = 4 & \quad \square & \quad 2 \text{ ways}. \\

n = 5 & & 5 \text{ ways}. \\
\end{align*}
n = 6

14 ways
THEOREM (Euler, Segner, Goldbach / 1750's):

There are \( \frac{1}{n-1} \binom{2(n-2)}{n-2} \) ways to triangulate the \( n \)-sided polygon.

<table>
<thead>
<tr>
<th>( n )</th>
<th>( \frac{1}{n} \binom{2(n-2)}{n-2} )</th>
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</thead>
<tbody>
<tr>
<td>3</td>
<td>(empty product) = 1</td>
</tr>
<tr>
<td>4</td>
<td>( \frac{4}{2} = 2 )</td>
</tr>
<tr>
<td>5</td>
<td>( \frac{5 \cdot 6}{2 \cdot 3} = 5 )</td>
</tr>
<tr>
<td>6</td>
<td>( \frac{6 \cdot 7 \cdot 8}{2 \cdot 3 \cdot 4} = 14 )</td>
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THEOREM (RSW 2004):

The $d$-fold cyclically symmetric triangulations of a (regular) $n$-sided convex polygon are counted by plugging in a primitive complex $d^\text{th}$ root of unity for $q$ in this:

$$\frac{1}{[n-1]_q} \left[\begin{array}{c} 2(n-2) \\ n-2 \end{array} \right]_q = \frac{[n]_q [n+1]_q \cdots [2n-4]_q}{[2]_q [3]_q \cdots [n-2]_q}$$

$q$-Catalan
EXAMPLE: \( n = 6 \)

\[
\frac{1}{[n-1]} \sum_{n-2}^{2(n-2)} = \frac{[6]_q [7]_q [8]_q}{[2]_q [3]_q [4]_q}
\]

\[
= 1 + \frac{a}{6} + \frac{a^2}{6} + \frac{a^3}{6} + 2a + 2a^2 + 2a^3 + 2a^4 + 2a^5 + 2a^6 + a^7 + \frac{a^8}{6}
\]

\( q = 1 \)

\( q = -1 \)

\( q = e^{\frac{2\pi i}{3}} \)

\( q = e^{\frac{2\pi i}{6}} \)

\( = 14 \)

\( = 6 \)

\( = 2 \)

\( = 0 \)

All 14 of them, e.g.

1-fold

2-fold

3-fold

6-fold

(None of them)
The $q$-Catalan $\frac{1}{[n-1]_q} \left[ \frac{2(n-2)}{n-2} \right]_q$ has many properties and interpretations:

- It is again a polynomial in $q$, with nonnegative coefficients.
- It again has meaning in geometry and in representation theory.
- It again counts something: if we plug in $q = p^m$ a prime power:

\[
\left\{ \text{orbits of } \mathbb{F}_{q^{2n-3}} \times \mathbb{F}_{q^n-3} \text{ acting on } \mathbb{F}_{q^{n-3}} \right\}
\]
Thanks for coming!