q-Narayana and q-Kreweras numbers for Weyl groups

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Eric Sommers (U. Mass- Amherst)

The mathematics of Michelle Wachs
January 8, 2015

The 4 basic food groups in my grad school math diet

In alphabetical order:

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In alphabetical order:

- Björner

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- Wachs

On work by M. Wachs published by others?

From “Spectra of symmetrized shuffling operators” with F. Saliola and V. Welker:

7. Acknowledgements

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No, let’s talk instead about why her recent work is on the right q-Narayana numbers!

Some directions of Catalan generalization

arbitrary q, t

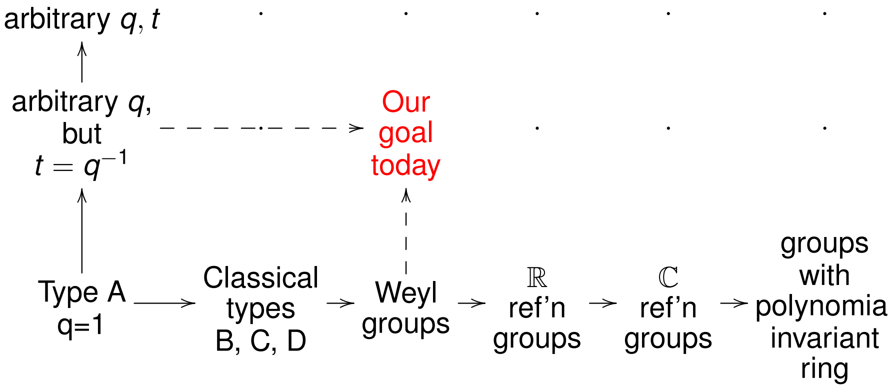
↑

arbitrary q, but $t = q^{-1}$

↑

Type A q=1 → Classical types A, B, C, D → Weyl groups → $\mathbb{R}$ ref'n groups → $\mathbb{C}$ ref'n groups → groups with polynomial invariant ring

Where we're headed



Outline

1. The numbers
   - The numbers in type A
   - Narayana numbers as h-vector
   - The definitions in all types

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2. The q-numbers
   - q-Catalans
   - q-Kreweras, q-Narayana
   - **Nilpotent** orbits
3. Properties
   - Principal-in-Levi orbits
   - Evaluations
   - The q-analogue of h-vector to f-vector

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4. Where do they come from ?
   - Springer fibers
   - A recursion of Shoji

Bell, Stirling, and unnamed numbers

Definition

Set partitions of $\{1, 2, \ldots, n\}$ are counted

- in total by Bell numbers $B(n)$,
- via number of blocks by Stirling numbers $S(n, k)$,
- via block size partition λ by unnamed numbers (?).

They have recurrences and generating functions,
but lack product formulas.

Bell, Stirling, and unnamed numbers

$S(4,1) = 1$ $\quad \lambda = (4) : 1$

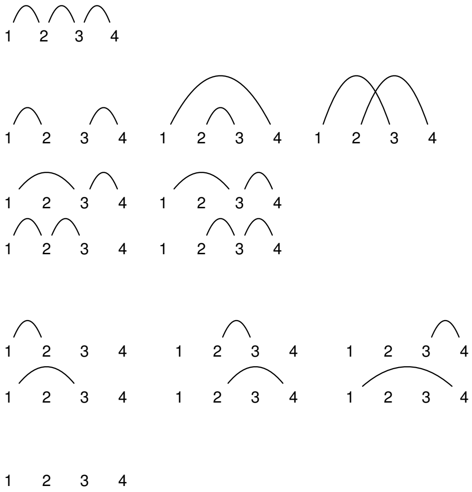
$S(4,2) = 7$ $\quad \lambda = (2^2) : 3$

$\lambda = (31) : 4$

$B(4) = 15$

$S(4,3) = 6$ $\quad \lambda = (21^2) : 6$

$S(4,1) = 1$ $\quad \lambda = (1^4) : 1$



The spoilsports ...

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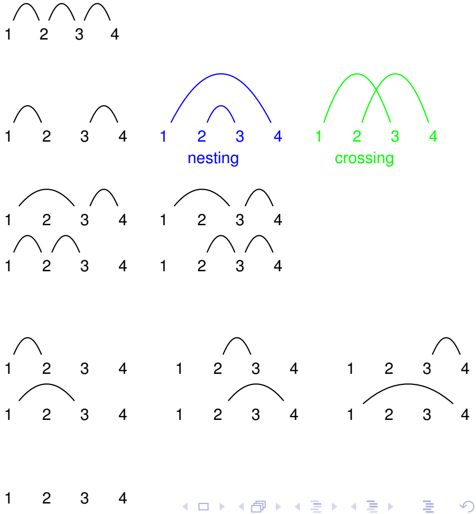

Catalan, Narayana, and Kreweras numbers

Definition

The noncrossing or nonnesting set partitions are counted

- in total by Catalan numbers $\mathrm{Cat}(n)$,
- via number of blocks by Narayana $N(n,k)$ numbers,
- via block size partition λ by Kreweras numbers $\mathrm{Krew}(\lambda)$.

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- via block size partition λ by Kreweras numbers $\mathrm{Krew}(\lambda)$.

They're better, IMHO.

Cat, Nar, Krew counting noncrossings

$N(4,1) = 1$ $\mathrm{Krew}(4) = 1$

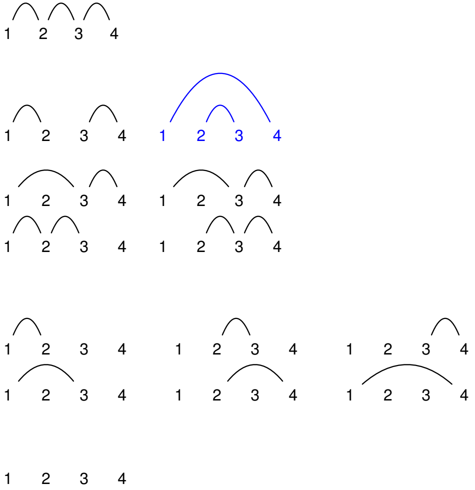
$N(4,2) = 6$ $\mathrm{Krew}(2^2) = 2$

$\mathrm{Krew}(31) = 4$

$\mathrm{Cat}(4) = 14$

$N(4,3) = 6$ $\mathrm{Krew}(21^2) = 6$

$N(4,1) = 1$ $\mathrm{Krew}(1^4) = 1$



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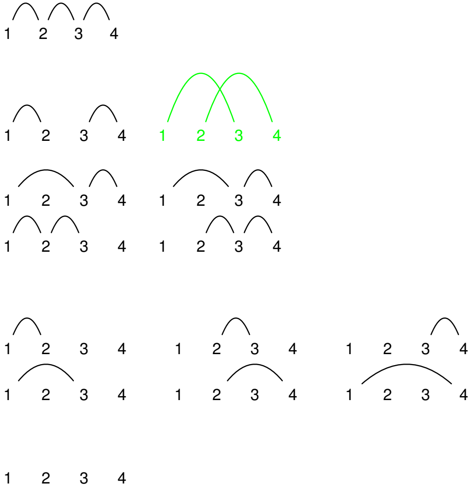

Catalan, Narayana, Kreweras formulas

They do have product formulas ...

Definition

$$\mathrm{Cat}(n) := \frac{1}{n+1}\binom{2n}{n}$$

$$N(n,k) := \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}$$

$$\mathrm{Krew}(\lambda) := \frac{1}{n+1}\binom{n+1}{\mu_1,\ldots,\mu_n} \text{ if } \lambda = 1^{\mu_1}2^{\mu_2}3^{\mu_3}\cdots \text{ partitions } n.$$

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Convention : $\binom{N}{\mu_1,\ldots,\mu_n} := \frac{N!}{\mu_1!\cdots\mu_n!(N-\sum_i \mu_i)!}$ if $\sum_i \mu_i \leq N$.

Kreweras sum to Narayana, which sum to Catalan

As one would expect, one can check these from the formulas:

Proposition

$$\mathrm{Cat}(n) = \sum_{k=1}^{n} N(n,k),$$

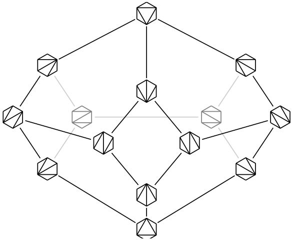
$$N(n,k) = \sum_{\substack{\text{partitions} \\ \lambda \text{ of } n: \\ \ell(\lambda)=k}} \mathrm{Krew}(\lambda)$$

where $\ell(\lambda) = \sum_i \mu_i$ *is the length or number of parts of* λ.

Narayana numbers as h-vector of the associahedron

Definition

The d-dimensional associahedron is a simple polytope with $(n+3)$-gon triangulations as vertices, diagonal flips as edges.



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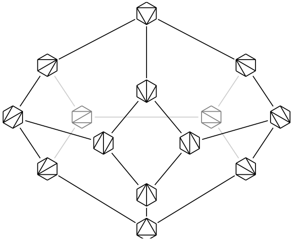

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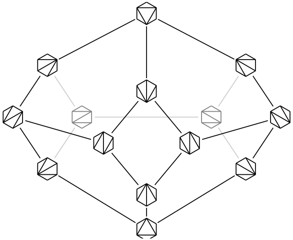

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$(h_0, h_1, h_2, h_3) = (1, 6, 6, 1)$

The h-vector to f-vector transformation

Definition

For P a d-dimensional simple polytope with f_i faces of dimension i, one can define the h-vector $(h_0, \ldots, h_d)$ via

$$\sum_{i=0}^{d} f_i t^i = \sum_{i=0}^{d} h_i (1+t)^i$$

$$\sum_{i=0}^{d} f_i (t-1)^i = \sum_{i=0}^{d} h_i t^i$$

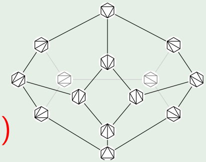
Narayana numbers as h-vector of the associahedron

Theorem (C. Lee 1989)

The Narayana numbers give the h-vector of the associahedron.

Example

The 3-dimensional associahedra has



$$(f_0, f_1, f_2, f_3) = (14, 21, 9, 1)$$
$$(h_0, h_1, h_2, h_3) = (1, 6, 6, 1)$$

$$14 + 21t + 9t^2 + 1t^3 = 1 + 6(1+t) + 6(1+t)^2 + 1(1+t)^3.$$

Quick review of W-noncrossing, nonnesting

Let $W \subset GL_\ell(\mathbb{R})$ be an irreducible finite reflection group.

Definition (Bessis, Brady-Watt, early 2000's)

The W-noncrossing partitions are

$$NC(W) := [e, c]_{\text{abs}}$$

Definition (Postnikov, mid-1990s)

The W-nonnesting partitions are

$$NN(W) := \text{Antichains}(\Phi^+)$$

W-Catalan counts W-noncrossing, nonnesting

Theorem

$$|NC(W)| = |NN(W)| = \mathrm{Cat}(W) := \prod_{i=1}^{\ell} \frac{e_i + h + 1}{e_i + 1}$$

where $(e_1, \ldots, e_\ell)$ are the exponents of the reflection hyperplane arrangement for W, and $h = \max\{e_i + 1\}$ is the Coxeter number, the order of any Coxeter element $c = s_1 \cdots s_\ell$ if the Coxeter system (W, S) has $S = \{s_1, \ldots, s_\ell\}$.

$\mathrm{Cat}(W)$ in type A

Example

Type A_{n-1} has $W = S_n$ acting on $\{x \in \mathbb{R}^n : \sum_i x_i = 0\}$.

One can choose $S = \{s_1, \ldots, s_{n-1}\}$ where $s_i = (i, i+1)$.

The exponents are $(1, 2, \ldots, n-1)$.

A choice of Coxeter element is $c = s_1 \cdots s_{n-1} = (1, 2, \ldots, n)$, an n-cycle, having order $h = n = \max\{2, 3, \ldots, n\}$.

$$\mathrm{Cat}(A_{n-1}) = \prod_{i=1}^{\ell} \frac{h + e_i + 1}{e_i + 1}$$

$$= \frac{(n+2)\cdot(n+3)\cdots(n+n)}{2\cdot 3\cdots n} = \frac{1}{n+1}\binom{2n}{n}.$$

W-Narayana, Kreweras

To elements of $NC(W)$ or $NN(W)$ one associates a hyperplane intersection subspace X, or parabolic subgroup W_X, having

- a rank (= codimension of X),
- a W-orbit $[X]$, or W-conjugacy class for W_X.

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Definition

The W-Narayana numbers $N(W, k)$ count the elements of $NC(W)$ or $NN(W)$ having a X of a fixed rank k.

They give the h-vector of the W-cluster complex or W-associahedron of Fomin-Zelevinsky 2003.

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Definition

The W-Kreweras numbers $\mathrm{Krew}(W, [X])$ count the elements of either $NC(W)$ or $NN(W)$ with a fixed W-orbit $[X]$.

Orlik-Solomon exponents give a product formula

Theorem (Broer, Douglass, Sommers, late 1990s)

$\mathrm{Krew}(W, [X])$ *has a product formula:*

$$\mathrm{Krew}(W, [X]) = \frac{1}{[N_W(W_X) : W_X]} \prod_{i=1}^{\ell} (h + 1 - e_i^X)$$

where $(e_1^X, \ldots, e_\ell^X)$ *are the Orlik-Solomon exponents of the reflection arrangement of* W *restricted to* X*.*

Fuss and rational generalization

Definition

Say m is very good for Φ if m is odd in types B, C, D, and if $\gcd(m,h) = 1$ in all other types, in which case define

$$\mathrm{Cat}(W,m) := \prod_{i=1}^{\ell} \frac{e_i + m}{e_i + 1}$$

$$\mathrm{Krew}(W,[X],m) := \frac{1}{[N_W(W_X) : W_X]} \prod_{i=1}^{\ell} (m - e_i^X)$$

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This captures the

- rational Catalan case $\gcd(m, n) = 1$ in type A_{n-1},
- W-Fuss-Catalan case $m = sh + 1$ in any type,
- and in particular, the usual W-Catalan case is $m = h + 1$

No problem q-ifying the W-Catalan

Definition

$$\mathrm{Cat}(W,q) := \prod_{i=1}^{\ell} \frac{[h+e_i+1]_q}{[e_i+1]_q}$$

where $[n]_q := 1 + q + q^2 + \cdots + q^{n-1}$.

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It's not silly, e.g., it satisfies a cyclic sieving phenomenon.

Theorem (Bessis-R. 2007)

For ζ a primitive h^{th} root of unity,

$$\mathrm{Cat}(W, q = \zeta^d)$$

counts elements of $NC(W) = [e,c]_{\mathrm{abs}}$ fixed conjugating by c^d.

And same for q-ifying $\mathrm{Cat}(W, m)$

Theorem

When m is very good, $\mathrm{Cat}(W, m; q) := \prod_{i=1}^{\ell} \frac{[e_i+m]_q}{[e_i+1]_q}$ *lies in* $\mathbb{N}[q]$.

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Very sketchy proof.

m is very good if and only if this formula

$$\chi(w) := \frac{\det(1 - q^m w)}{\det(1 - qw)}$$

is a genuine graded W-character:

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$\mathrm{Cat}(W, m; q)$ is its W-fixed space $(S/(\theta))^W$ Hilbert series. □

A_{n-1} q-Narayanas in Wachs' IMA talk 11/12/2014 ...

$$N(A_{n-1}, j, q) := \frac{q^{j(j+1)}}{[n]_q} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j+1 \end{bmatrix}_q$$

q-Narayana polynomials

The Narayana numbers have a closed form formula

$$N_n(t) = \sum_{j=0}^{n-1} \frac{1}{n} \binom{n}{j} \binom{n}{j+1} t^j.$$

Recall that the Narayana numbers refine the Catalan numbers

$$N_n(1) = C_n.$$

The Fürlinger-Hofbauer q-Narayana polynomials are defined by

$$N_n(q, t) := \sum_{j=0}^{n-1} q^{j(j+1)} \frac{1}{[n]_q} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j+1 \end{bmatrix}_q t^j.$$

... and type B q-Narayanas came later in her talk ...

$$N(B_n, j, q) := (q^2)^{j^2} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2}$$

Super q-Narayana polynomials (Krattenthaler and MW)

For $n \geq s$, define the super q-Narayana polynomials

$$N_n^{(s)}(q,t) := \begin{bmatrix} 2s \\ s \end{bmatrix}_q \sum_{j=0}^{n-s} q^{j(j+1)} \begin{bmatrix} n \\ s \end{bmatrix}_q^{-1} \begin{bmatrix} n \\ j \end{bmatrix}_q \begin{bmatrix} n \\ j+s \end{bmatrix}_q t^j.$$

Note $N_n^{(1)}(q,t) = (1+q)N_n(q,t)$.

$N_n^{(0)}(1,t)$ is the type B Narayana polynomial.

Gessel proved $N_n^{(s)}(1,t) \in \mathbb{N}[t]$ by deriving a γ-positivity formula.

Several questions arise

Question

- *Are there q-Kreweras polynomials of types A, B, C, D? All types? Do they sum to $\mathrm{Cat}(W, q)$?*
- *In types A, B do they sum to the above q-Narayanas?*

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Answer

Sommers' work answers yes to 1st question for Weyl groups, if we associate a q-Kreweras number to each nilpotent orbit.

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Actually, yes to all above, but we don't understand it uniformly!

What parametrizes a q-Kreweras number?

We won't just get a q-Kreweras number for each W-orbit $[X]$ of intersection subspace. Instead we will get

$$\mathrm{Krew}(e, m, q)$$

for each ...

- Weyl group W, with a root system Φ, and
- a nilpotent orbit e in its Lie algebra $\mathfrak{g}$, and
- a positive integer m which is very good for Φ.

Type A nilpotent orbits

In type A_{n-1}, $G = SL_n(\mathbb{C})$ conjugates $\mathfrak{g} = sl_n(\mathbb{C}) = \mathbb{C}^{n \times n}$, and nilpotent orbits are represented by Jordan canonical forms, parametrized by partitions λ of n.

Example

In $sl_8(\mathbb{C})$, the partition $\lambda = 32^21$ corresponds to the $SL_8(\mathbb{C})$-orbit of

$$\begin{bmatrix} 0 & 1 & 0 & & & & & \\ & 0 & 1 & & & & & \\ & & 0 & & & & & \\ & & & 0 & 1 & & & \\ & & & & 0 & & & \\ & & & & & 0 & 1 & \\ & & & & & & 0 & \\ & & & & & & & 0 \end{bmatrix}$$

Type A q-Kreweras formula

In type A_{n-1}, very good for m means $\gcd(m, n) = 1$.

Theorem

For partitions $\lambda = 1^{\mu_1} 2^{\mu_2} 3^{\mu_3} \cdots$ *of* n *with* $\gcd(m, n) = 1$,

$$\mathrm{Krew}(e_\lambda, m; q) = q^{m(n-\ell(\lambda))-c(\lambda)} \frac{1}{[m]_q} \begin{bmatrix} m \\ \mu_1, \ldots, \mu_n \end{bmatrix}_q .$$

where

$$c(\lambda) := \sum_j \lambda'_j \lambda'_{j+1}, \text{ with } \lambda' \text{ the transpose partition to } \lambda$$

$$\begin{bmatrix} m \\ \mu \end{bmatrix}_q := \frac{[m]!_q}{[\mu_1]!_q \cdots [\mu_\ell]!_q [m - \sum_i \mu_i]!_q}$$

Types $B/C/D$

Φ	$\mathfrak{g}$	Condition on $\lambda = 1^{\mu_1} 2^{\mu_2} 3^{\mu_2} \ldots$ parametrizing nilpotent orbits
B_n	so_{2n+1}	$\lvert\lambda\rvert = 2n + 1$, and μ_j even for j even
C_n	sp_{2n}	$\lvert\lambda\rvert = 2n$, and μ_j even for j odd
D_n	so_{2n}	$\lvert\lambda\rvert = 2n$, and μ_j even for j even

A slight lie in type D_n: these are O_{2n} orbits on so_{2n}, not SO_{2n}-orbits, leading to an extra factor of 2 in some formulas.

Type B, C q-Kreweras formulas– the gestalt picture

Introduce notations

$$\hat{N} := \lfloor N/2 \rfloor,$$
$$\hat{\mu} := (\lfloor \mu_1/2 \rfloor, \lfloor \mu_2/2 \rfloor, \ldots) \text{ if } \mu = (\mu_1, \mu_2, \ldots).$$

Theorem

For $\lambda = 1^{\mu_1} 2^{\mu_2} 3^{\mu_3} \ldots$ a type B_n or type C_n partition, and m odd,

$$\mathrm{Krew}(e_\lambda, m; q) = q^{\exp(\lambda, m) + \epsilon} \begin{bmatrix} \hat{m} - \hat{L}(\lambda) \\ \hat{\mu} \end{bmatrix}_{q^2} \cdot \prod_{i=1}^{\hat{L}(\lambda)} (q^{m-2i+1} - 1)$$

What was that power $q^{\exp(\lambda,m)+\epsilon}$ in front?

$$\epsilon := \begin{cases} \frac{1}{4} & \text{in type } B_n, \\ 0 & \text{in type } C_n \text{ for } \ell(\lambda) \text{ even}, \\ \frac{1}{4} - \frac{\ell(\lambda)}{2} & \text{in type } C_n \text{ for } \ell(\lambda) \text{ odd.} \end{cases}$$

and

$$\exp(\lambda, m) := m(n - \hat{\ell}(\lambda)) - \frac{c(\lambda)}{2} + \tau(\lambda) - \frac{L(\lambda)}{4}$$

with

$$L(\lambda) := |\{i : \mu_i \text{ odd}\}|$$

$$\tau(\lambda) := \frac{1}{2} \sum_{\substack{j \not\equiv |\lambda| \bmod 2 \\ \mu_j \text{ even}}} \mu_j$$

Type D q-Kreweras formulas

Here μ_1 plays a special role. Define $\mu_{\geq 2} := (\mu_2, \mu_3, \ldots)$.

Theorem

For m odd and λ a type D_n partition,
$\mathrm{Krew}(e_\lambda, m; q)$ *is* $q^{\exp(\lambda,m)}$ *times these:*

$$\begin{cases} q^{m-\frac{\ell(\lambda)}{2}+1} \begin{bmatrix} \hat{m}-(\hat{L}(\lambda)-1) \\ \hat{\mu} \end{bmatrix}_{q^2} \cdot \prod_{i=1}^{\hat{L}(\lambda)-1} (q^{m-2i+1}-1) & \text{if } \mu_1 \text{ odd}, \\ q^{\frac{\ell(\lambda)}{2}-\mu_1(\lambda)} \begin{bmatrix} \hat{m}-\hat{L}(\lambda) \\ \hat{\mu}_{\geq 2} \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+1-\hat{L}(\lambda)-|\hat{\mu}_{\geq 2}| \\ \hat{\mu}_1 \end{bmatrix}_{q^2} \cdot \prod_{i=1}^{\hat{L}(\lambda)} (q^{m-2i+1}-1) & \text{if } \mu_1 \text{ even, some } \mu_j \text{ odd}, \\ q^{\frac{\ell(\lambda)}{2}-\tau(\lambda)} \begin{bmatrix} \hat{m} \\ \hat{\mu} \end{bmatrix}_{q^2} + q^{\frac{\ell(\lambda)}{2}-\mu_1} \begin{bmatrix} \hat{m} \\ \hat{\mu}_{\geq 2} \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+1-|\hat{\mu}_{\geq 2}| \\ \hat{\mu}_1 \end{bmatrix}_{q^2} & \text{if } \mu_j \text{ all even}. \end{cases}$$

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(Thanks, Ted Cruz!)

Defining the q-Narayana numbers in general

Later we define a mysterious statistic $\kappa(e)$ on nilpotent orbits e.

Example

Φ	$\kappa(e_\lambda)$
A_{n-1}	$\ell(\lambda)$
B_n/C_n	$\hat{\ell}(\lambda)$
D_n	$\begin{cases} \hat{\ell}(\lambda) & \text{if } \mu_1 \text{ is even,} \\ \hat{\ell}(\lambda) - 1 & \text{if } \mu_1 \text{ is odd.} \end{cases}$

Definition

Given m very good for Φ and $0 \leq k \leq \ell$, define

$$\mathrm{Nar}(\Phi, m, k; q) := \sum_{e : \kappa(e) = k} \mathrm{Krew}(e, m; q).$$

Type A, B, C q-Narayanas

Theorem

The q-Narayana numbers in types A, B/C are ...

Φ	$\mathrm{Nar}(\Phi, m, k; q)$
A_{n-1}	$q^{(n-1-k)(m-1-k)} \frac{1}{[k+1]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} m-1 \\ k \end{bmatrix}_q$
B_n/C_n	$(q^2)^{(n-k)(\hat{m}-k)} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m} \\ k \end{bmatrix}_{q^2}$

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Its not hard to see that they lie in $\mathbb{N}[q]$.

At $m = h + 1$ they give the q-Narayanas used by Wachs.

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Its not hard to see that they lie in $\mathbb{N}[q]$.

At $m = h + 1$ they give the q-Narayanas used by Wachs.

Question

Even at $q = 1$, do they relate to work of Friedman-Stanley?

But who are the type D q-Narayana's?

The type D q-Narayana numbers are q-analogues of these:

$$[Nar(D_n, m, k; q)]_{q=1} = \binom{\hat{m}}{k}\binom{n}{k} + \binom{\hat{m}+1}{k}\binom{n-2}{k-2}$$

We only know simple formulas (not sums) for $Nar(D_n, m, k; q)$ when $k = 0, 1, n-1, n$.

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We only know simple formulas (not sums) for $Nar(D_n, m, k; q)$ when $k = 0, 1, n-1, n$. The formulas are consistent with this:

Conjecture

If m is very good for Φ, then $\mathrm{Nar}(\Phi, m, k; q)$ lies in $\mathbb{N}[q]$.

Problem

Find simple formulas for all $Nar(D_n, m, k; q)$ making this clear.

Regular-in-a-Levi nilpotent orbits

Various divisibility and evaluation properties of the q-Kreweras numbers relate to a special subclass of nilpotent orbits.

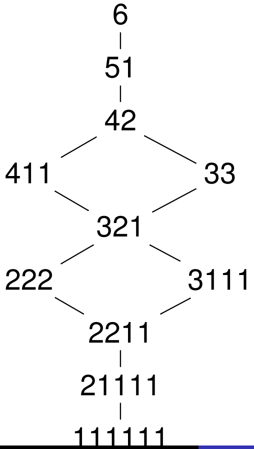
Definition

For a W-orbit $[X]$ of intersection subspaces X, let e_X be the G-orbit in $\mathfrak{g}$ of the principal nilpotent in the Levi subalgebra $\mathfrak{g}_X$

$$\begin{array}{ccc} \begin{array}{c} W\text{-conjugacy classes of} \\ \text{parabolic subgroups} \end{array} & & \\ \updownarrow & & \\ \begin{array}{c} W\text{-orbits of} \\ \text{intersection subspaces} \end{array} & \hookrightarrow & \begin{array}{l} \text{nilpotent} \\ G\text{-orbits in } \mathfrak{g} \end{array} \\ & & \\ [X] & \mapsto & e_X \end{array}$$

All nilpotent orbits in type A are principal-in-Levi

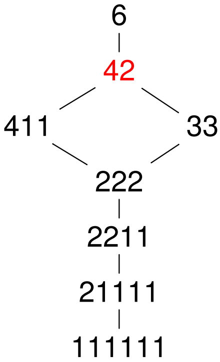
Type A_5
$\mathfrak{g} = sl_6$
$W = S_6$



$e_\lambda \leftrightarrow S_{\lambda_1} \times S_{\lambda_2} \times \cdots$

Type B/C principal-in-Levi means at most one μ_i odd

Type C_3
$\mathfrak{g} = sp_6$
$W = B_3$



Their corresponding paraboblic subgroups $W_X \leq B_3$

$$6 = B_3$$

$$42$$

$$411 = S_1 \times B_2 \qquad 33 = S_3$$

$$222 = S_2 \times B_1$$

$$2211 = S_2 \times S_1$$

$$21111 = S_1 \times S_1 \times B_1$$

$$111111 = S_1 \times S_1 \times S_1$$

Evaluating q-Kreweras, q-Narayanas at $q = 1$

Theorem

Let m be very good for Φ.
For e_X principal-in-a-Levi, $\mathrm{Krew}(\Phi, e, m; q)$ *lies in* $\mathbb{N}[q]$,

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$$[\mathrm{Krew}(\Phi, e_X, m; q)]_{q=1} = \mathrm{Krew}(W, [X], m)$$

Also $\kappa(e_X) = \dim(X)$ when e_X is principal-in-Levi, implying this:

Corollary

$$[\mathrm{Nar}(\Phi, m, k; q)]_{q=1} = \sum_{[X]:\dim(X)=k} \mathrm{Krew}(W, [X], m)$$
$$= \mathrm{Nar}(W, m, k).$$

What about the not principal-in-Levi's at $q = 1$?

Theorem

Let m be very good for Φ.
For e not principal-in-a-Levi,

- $\mathrm{Krew}(\Phi, e, m; q)$ *vanishes at $q = 1$, and*
- *is furthermore divisible by $q^{m-1} - 1$.*

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Question

What do $(m-1)^{st}$ root-of-unity evaluations, besides $q = 1$, mean for $\mathrm{Krew}(\Phi, e_X, m; q)$ when e_X is principal-in-Levi?

A cyclic sieving phenomenon (CSP)

We know for the Fuss-Catalan very good values $m = sh + 1$.

Definition (Armstrong 2006)

The W-generalization of s-divisible noncrossing partitions is

$$NC^{(s)}(W) := \{s\text{-multichains}\, w_1 \leq \cdots \leq w_s \text{ in } NC(W)\}.$$

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Conjecture

Let $m = sh + 1$ and $\zeta := e^{\frac{2\pi i}{m-1}}$. When e_X is in principal-in-Levi,

$$[\mathrm{Krew}(\Phi, e_X, m; q)]_{q=\zeta^d}$$

counts elements of $NC^{(s)}(W)$ with V^{w_1} in $[X]$, fixed by c^d.

At least in all the classical types

Theorem

The CSP conjecture holds in classical types A, B, C, D: for e_X principal-in-Levi, $[\mathrm{Krew}(\Phi, e_X, m; q)]_{q=\zeta^d}$ counts the elements of $NC^{(s)}(W)$ having V^{w_1} in $[X]$ that are fixed by c^d.

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Proof.

Bad: compare the $q = \zeta^d$ evaluation to known counts.

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Proof.

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In type A, it was (pretty much) known; types B, C, D are new.

In type D, the case structure is very intricate, a testament to the "correctness" of the formulas for the q-Kreweras!

What's the q-analogue of the f-vector?

Finite cluster complexes do have a q-analogue of the f-vector.

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Recall when m is very good for Φ, graded W-rep'n $S/(\theta)$ has

$$\begin{aligned}
\mathrm{Cat}(W,m) &= \dim_{\mathbb{C}} (S/(\theta))^W &&= \langle \wedge^0 V, S/(\theta)\rangle \\
\mathrm{Cat}(W,m,q) &= \mathrm{Hilb}\left((S/(\theta))^W, q\right) &&= \textstyle\sum_i \langle \wedge^0 V, S/(\theta)_i\rangle q^i.
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Theorem (Armstrong-Rhoades-R. 2014)

The cluster complex of type Φ has $f_k = f_k(W, h+1)$ where

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Definition

$$f_k(W,m;q) := \sum_i \langle \wedge^k V, S/(\theta)_i \rangle \, q^i$$

The q-analogue of f-vectors in classical types

In types $A, B/C, D$, Gyoja, Nishiyama, Shimura 1999 give $f_k(W,m;q)$ for m very good, not just $m = h+1$.

Φ	$f_k(W,m;q)$
A_{n-1}	$q^{\binom{k+1}{2}} \frac{1}{[m]_q} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \begin{bmatrix} m+n-k-1 \\ n \end{bmatrix}_q$
B_n/C_n	$q^{k^2} \begin{bmatrix} \hat{m} \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+n-k \\ \hat{m} \end{bmatrix}_{q^2}$
D_n	$q^{k^2} \begin{bmatrix} \hat{m} \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+n-k \\ \hat{m} \end{bmatrix}_{q^2} + q^{n-2k+k^2} \begin{bmatrix} \hat{m}+1 \\ k \end{bmatrix}_{q^2} \begin{bmatrix} \hat{m}+n-k-1 \\ \hat{m}-1 \end{bmatrix}_{q^2}$

A q-analogue of h-to-f-vector

Thus the usual cluster complex h-to-f-vector identity would be

$$\sum_k f_k(W, h+1)t^k = \sum_k \mathrm{Nar}(W, h+1, k)\,(1+t)^k$$

Theorem

$$\sum_k f_k(A_{n-1}, m; q)t^k = \sum_k \mathrm{Nar}(A_{n-1}, m, k; q)\,(-tq; q)_k,$$
$$\sum_k f_k(B_n/C_n, m; q)t^k = \sum_k \mathrm{Nar}(B_n/C_n, m, k; q)(-tq; q^2)_k.$$

where $(x; q)_k = (1-x)(1-qx)\cdots(1-q^{k-1}x)$,
so that $(-tq; q^r)_k$ is a q-analogue of $(1+t)^k$.

A q-analogue of h-to-f-vector

The previous type $A, B/C$ identities are both special cases of a ${}_2\phi_1$-transformation of Jackson:

$$ {}_2\phi_1 \left[\begin{matrix} q^{-N} & b \\ - & c \end{matrix} \middle| q, z \right] = \frac{(c/b;q)_N}{(c;q)_N} {}_3\phi_2 \left[\begin{matrix} q^{-N} & b & bzq^{-N}/c \\ - & bq^{1-N}/c & 0 \end{matrix} \middle| q, q \right] $$

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(Thanks, Dennis Stanton!)

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However, they are also both instances of the following.

Theorem

When m is very good for Φ,

$$\sum_{k=0}^{\ell} f_k(\Phi, m, k; q) t^k = \sum_{k=0}^{\ell} \mathrm{Something}_k(q, t)$$

for a fairly explicit product $\mathrm{Something}(W, m, k; q, t)$,

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- $\mathrm{Nar}(A_{n-1}, m, k; q)(-tq; q)_k$ for $\Phi = A_{n-1}$,
- $\mathrm{Nar}(B_n/C_n, m, k; q)(-tq; q^2)_k$ for $\Phi = B_n/C_n$.

Remember Springer fibers?

Consider the nilcone

$$\mathcal{O} := \{\text{all nilpotent elements } e \text{ in } \mathfrak{g}\}$$

which is a singular variety inside $\mathfrak{g}$.

T. Springer's desingularized it using the flag manifold

$$G/B \cong \mathcal{B} = \{\text{all Borel subalgebras } \mathfrak{b} \text{ in } \mathfrak{g}\}$$

by creating this space

$$\tilde{\mathcal{O}} := \{(e, \mathfrak{b}) \in \mathcal{O} \times G/B : [e, \mathfrak{b}] \subset \mathfrak{b}\}.$$

with its two coordinate projection maps:

$$\mathcal{O} \xleftarrow{\pi_1} \tilde{\mathcal{O}} \xrightarrow{\pi_2} \mathcal{B}$$

The boring fiber shows it's smooth

$$\mathcal{O} \xleftarrow{\pi_1} \tilde{\mathcal{O}} \xrightarrow{\pi_2} \mathcal{B}$$

The projection π_2 has as typical fiber an affine space

$$\pi_2^{-1}(\mathfrak{b}_+) = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}_\alpha \cong \mathbb{C}^{|\Phi_+|}$$

Corollary

The total space $\tilde{\mathcal{O}}$ is smooth.

Proof.

The base $\mathcal{B} = G/B$ is smooth, the fiber is affine. □

The Springer fiber is interesting

The Springer fibers are the fibers of the other projection π_1:

$$\mathcal{B}_e := \pi_1^{-1}(e) = \{\mathfrak{b} \in G/B : [e, \mathfrak{b}] \subset \mathfrak{b}\}$$

Their cohomology $H^*(\mathcal{B}_e)$ has an interesting graded W-action.

Example

In type A, the ring $H^*(\mathcal{B}_{e_\mu})$, sometimes called R_μ, has its graded S_n-Frobenius characteristic given by the modified Hall-Littewood symmetric function $q^{n(\mu)} H_\mu(\mathbf{x}; q^{-1})$.

Shoji's recursion

Shoji 1982 gave an identity that recursively determines the graded W-characters $H^*(\mathcal{B}_e)$. Its coefficients involve

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$$A(e) := Z_G(e)/Z_G^0(e)$$

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This lets one refine the graded W-representations

$$H^*(\mathcal{B}_e) = \bigoplus_{\phi} H^*(\mathcal{B}_e)^{\phi}$$

into $A(e)$-isotypic components for $A(e)$-irreducibles ϕ.

Sommers's reformulation: the rough idea

Sommers recast Shoji's recursion in terms of W-irreducibles χ:

$$H^*(\mathcal{B}) \otimes \chi = \sum_e \sum_\phi \alpha(e, \phi, \chi, q) H^*(\mathcal{B}_e)^\phi. \tag{1}$$

Sommers's reformulation: the rough idea

Sommers recast Shoji's recursion in terms of W-irreducibles χ:

$$H^*(\mathcal{B}) \otimes \chi = \sum_e \sum_\phi \alpha(e, \phi, \chi, q) H^*(\mathcal{B}_e)^\phi. \qquad (1)$$

One can restate the graded character formula for m very good,

$$\chi_{S/(\theta)}(w; q) = \det(1 - q^m w) / \det(1 - qw),$$

as saying $\quad S/(\theta) = \sum_{k=0}^{\ell} (-q^m)^k S \otimes \wedge^k V.$

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Then using $H^*(\mathcal{B}) \cong S/(S_+^W)$, and (1) at $\chi = \wedge^k V$, summed over $k = 0, 1, \ldots, \ell$, Sommers proved a key result...

How to define q-Kreweras using Sommers's result

Theorem (Sommers 2011)

$$S/(\theta) = \sum_{e} \sum_{\phi} f(e, \phi, m; q) H^*(\mathcal{B}_e)^{\phi}.$$

This was the starting point for everything, such as ...

Definition

$$\mathrm{Krew}(\Phi, e, m; q) := f(e, \mathbf{1}_{A(e)}, m; q)$$

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For example, it immediately implies

$$\mathrm{Cat}(W, m; q) = \sum_e \mathrm{Krew}(\Phi, e, m; q)$$

since the W-rep $\mathbf{1}_W$ appears only in $H^0(\mathcal{B}, e) = H^0(\mathcal{B}, e)^{\mathbf{1}_{A(e)}}$.

How to define the q-Narayana statistic $\kappa(e)$

Recall there was a mysterious statistic $\kappa(e)$ used in defining

$$\mathrm{Nar}(\Phi, m, k; q) := \sum_{e:\kappa(e)=k} \mathrm{Krew}(\Phi, e, m; q)$$

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$\kappa(e) := \langle V, H^*(\mathcal{B}_e)\rangle$, the multiplicity of V in $H^*(\mathcal{B}_e)$.

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This definition works extremely well, as

- $\kappa(e) = \dim(X)$ when $e = e_X$ is principal-in-a-Levi,
- for almost all nilpotent orbits e, knowing within $H^*(\mathcal{B}_e)$ where V occurs (degrees, $A(e)$-isotypic components) determines via a simple product formula where all other $\wedge^k V$ occur, by another result of Sommers 2011.

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Other properties of the $f(e, \phi, m; q)$

- They lie in $\mathbb{Z}[q]$.
- At $q = 1$, they vanish unless $e = e_X$ is principal-in-Levi, in which case for every ϕ they have value $\mathrm{Krew}(W, [X], m)$.
- They can be computed via cardinalities of nilpotent orbits over $\mathbb{F}_q$, together with (available!) info about the W-representations $H^*(\mathcal{B}, e)$.

Thanks

Thanks for listening,

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Thanks for listening,

and thank you, Michelle, for having taught us so much!