Whitney numbers for poset cones
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• Zaslavsky's Theorems
  counting chambers in hyperplane arrangements and cones

• Braid arrangements, poset cones, and linear extensions

• Two formulas for any poset

• Foata's thesis and disjoint unions of chains
Zaslavsky's Theorems

\[ A = \{H_1, H_2, \ldots, H_N\} \]

an arrangement of hyperplanes in \( V = \mathbb{R}^n \)

\( \mathbb{R}^n \)

dissects the complement \( V \setminus A \) into connected components called chambers

10 chambers

How to count them?
More generally, a cone $K$ in $A$ is any intersection of its (open) halfspaces, containing a subset of the chambers of $A$.

How to count them?

$K_1$: 5 chambers inside cone $K_1$

$K_2$: 3 chambers inside cone $K_2$
Introduce the poset of intersections $L(\mathcal{A}) := \{\text{intersection subspaces}\} \quad X = H_{i_1} \cap H_{i_2} \cap \ldots \cap H_{i_k}$ ordered via reverse inclusion.

and for the cones $\mathcal{K}$, the subposet of interior intersections

$$\text{int}(\mathcal{K}) = \{X \in L(\mathcal{A}) : X \cap \mathcal{K} \neq \emptyset\}$$
To count chambers, label $X \in \mathcal{L}(A)$ by Möbius function values $\mu(v,x)$

\[
\begin{array}{c}
\mu(-1) \quad \mu(-1) \quad \mu(-1) \\
H_1 \quad H_2 \quad H_3 \quad H_4 \quad H_5 \\
1 \quad 1 \\
\end{array}
\]

\[
\begin{array}{c}
\mu(0) = 4 \\
\mu(1) = 5 \\
\mu(2) = 1 \\
\end{array}
\]

\[
\mathbf{THEOREM \ (Zaslavsky \ 1979)}
\]

**# chambers of $A = \sum_{X \in \mathcal{L}(A)} |\mu(v,x)| = c_0 + c_1 + \ldots + c_n \quad \text{where} \quad c_k = \sum_{X \in \mathcal{L}(A) : \text{codim}(x) = k} |\mu(v,x)| \quad \text{$k$th Whitney number of $A$}**

\[
\begin{array}{c}
\text{# chambers} \\
10 = 1 + 5 + 4 = c_0 + c_1 + c_2 \\
\end{array}
\]
More generally...

**THEOREM (Zaslavsky 1977)** For any cone $K$ in $A$,

$$\text{# chambers of } A = \sum_{x \in \text{int}(K)} |\mu(v,x)| = c_0(K) + \ldots + c_n(K)$$

where $c_k(K) = \sum_{x \in \text{int}(A), \text{codim}(x) = k} |\mu(v,x)|$.

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Diagram 1:

$H_2 \sim H_3 \sim H_4 \sim H_5$

$V$

$c_0 = 1$

$c_0 + c_1 = 1 + 4 + 0$

= 5 chambers

Diagram 2:

$H_3 \sim H_4$

$V$

$c_0 = 1$

$c_0 + c_1 + c_2 = 1 + 2 + 0$

= 3 chambers
Define the generating function

\[ \text{Poin}(t,t) = c_0 + c_1 t + c_2 t^2 + \ldots + c_n t^n \]

Poincaré polynomial of \( \Lambda \)

It gets its name from the interpretation

\[ c_k = \text{rank} \bigg[ H_k \left( \mathbb{C}^n \setminus \Lambda, \mathbb{Z} \right) \bigg] \]

called the complexified complement of \( \Lambda \)

For any cone \( K \) in \( A \), we'll similarly call

\[ \text{Poin}(K,t) = c_0(K) + c_1(K) t + c_2(K) t^2 + \ldots + c_n(K) t^n \]

the Poincaré polynomial of \( K \).

**GOAL:**

Interpret \( \text{Poin}(K,t) \) combinatorially, whenever possible.
Braid arrangements
(the motivating example)

The braid arrangement in $\mathbb{R}^n$ has hyperplanes $H_{ij} = \{ x_i = x_j \}$ for $1 \leq i < j \leq n$ in $\mathbb{R}^3$ intersect with $[1]^n$.

Inside the braid arrangement,

\[
\text{chambers} \leftrightarrow \text{permutations } \sigma = (\sigma_1, \ldots, \sigma_n) \text{ in } S_n
\]
intersection subspaces $X_{\pi}$

set partitions $\pi=\{B_1, B_2, \ldots, B_k\}$ of $\{1, 2, \ldots, n\} = \bigcup_{i=1}^{k} B_i$
A = braid arrangement in \( \mathbb{R}^n \) has many expressions for its Poincaré polynomial:

\[
\text{Poin}(A, t) = (1 + t)(1 + 2t)(1 + 3t) \cdots (1 + (n-1)t)
\]

\[
= \sum_{\sigma \in S_n} t^{n - \#\text{cycles(\sigma)}}
\]

\[
= \sum_{\sigma \in S_n} t^{n - \#\text{LRmax(\sigma)}}
\]

where \( \text{LRmax}(\sigma) = \text{left-to-right maxima of } \sigma \)

e.g. \( \sigma = 418253697 \) has \( \#\text{LRmax}(\sigma) = 3 \)
Inside the braid arrangement,

\[ \text{cones } K_p \leftrightarrow \text{ posets } P \text{ on } \{1,2,\ldots,n\} \]

\[ K_p = \bigcap_{i < j} \{ x_i < x_j \} \]

\[ \text{e.g. } K_p = \{ x_1 < x_2 \} \cap \{ x_3 < x_4 \} \leftrightarrow P = \{ 2 \overset{4}{\longrightarrow} 3 \overset{4}{\longleftarrow} \} \]

chambers inside \( K_p \leftrightarrow \) linear extensions

\[ \sigma = (\sigma_1 < \sigma_2 < \ldots < \sigma_n) \text{ of } P \]

\[ \Rightarrow \text{ LinExt}(P) \]

\[ \text{e.g. } P = \{ 2 \overset{4}{\longrightarrow} 3 \overset{4}{\longleftarrow} 4 \} \]

\[ \text{LinExt}(P) = \{ 1234, 1324, 1342, 3124, 3142, 3412 \} \]
Thus Zaslavsky's Theorem for cones gives

**COROLLARY**

\[
\# \text{LinExt}(P) = c_0(K_p) + c_1(K_p) + \ldots + c_n(K_p) = \left[ \text{Poin}(K_p,t) \right]_{t=1}
\]

#Hard to compute for arbitrary posets \( P \)

[Brightwell-Winkler] 1991

**PROBLEM:**

Interpret \( \text{Poin}(K_p,t) \) for posets \( P \), by refining the count \( \# \text{LinExt}(P) \).
\begin{align*}
\text{EXAMPLE} & \quad P = \begin{array}{c} 2 \\ 1 \\ 4 \\ 3 \end{array} \\
\text{Lin}\text{Extn} & = \text{chambers} = 1 + 4 + 1 = 6
\end{align*}
PROBLEM: Interpret $\text{Poin}(K_p,t)$ for posets $P$, by refining the count $\# \text{LinExt}(P)$.

We had three solutions for $P=\circ \circ \ldots \circ$

where

$$\text{LinExt}(P)=S_n$$

$$\# \text{LinExt}(P)=n!$$

$$\text{Poin}(K_p,t) = (1+t)(1+2t)(1+3t)\ldots(1+(n-1)t)$$

$$= \sum_{\sigma \in S_n} t^{n - \#\text{cycles}(\sigma)}$$

$$= \sum_{\sigma \in S_n} t^{n - \#\text{LRmax}(\sigma)}$$
Two formulas for any poset

**THEOREM (Porpalen-Barry-Kim-R. 2019)**

\[ \text{Poin}(K_P,t) = \sum_{\sigma \in S_n} t^{n - \# \text{cycles}(\sigma)} \]

**DEFINITION:**
- cycles of \( \sigma \) are antichains in \( P \)
- the quotient pre-poset \( P/\sigma \) collapses no strict order relations \( i <_P j \)

\[ C_2 = 1 \]
\[ C_1 = 4 \]
\[ C_0 = 1 \]
What some of those quotient pre-posets look like:

\[ a \rightarrow 4 \]
\[ 1 \rightarrow 3 \]
\[ 24 \rightarrow \]
\[ 1 \rightarrow 13 \]

\[ a \rightarrow 4 \]
\[ 1 \rightarrow 3 \]
\[ 24 \rightarrow \]
\[ 1 \rightarrow 13 \]

\[ a \rightarrow 4 \]
\[ 1 \rightarrow 3 \]
\[ 24 \rightarrow \]
\[ 1 \rightarrow 13 \]

\[ a \rightarrow 4 \]
\[ 1 \rightarrow 3 \]
\[ 24 \rightarrow \]
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\[ a \rightarrow 4 \]
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\[ 1 \rightarrow 13 \]

\[ a \rightarrow 4 \]
\[ 1 \rightarrow 3 \]
\[ 24 \rightarrow \]
\[ 1 \rightarrow 13 \]

P-transverse \( \overset{\text{def}}{=} \left\{ \begin{array}{l}
  \bullet \text{cycles of } \sigma \text{ are antichains in } P \\
  \bullet \text{the quotient pre-poset } P/\sigma \\
  \text{collapses no strict order relations } i<_P j
\end{array} \right. \)
Note the summation in

\[ \text{Poin}(\mathcal{P}_P, t) = \sum_{\sigma \in S_n} t^{n - \#\text{cycles}(\sigma)} \]

is not over LinExt(P), unlike here:

**THEOREM (Dorpalen-Barry - Kim - R. 2019)**

\[ \text{Poin}(\mathcal{P}_P, t) = \sum_{\sigma \in \text{LinExt}(P)} t^{n - \#P-LRmax(\sigma)} \]

where \( P-LRmax(\sigma) \) generalizes \( LRmax(\sigma) \)

(in an interesting way, not described here)

The proof is a bijection

\[
\begin{array}{ccc}
\text{LinExt}(P) & \rightarrow & P\text{-transverse permutations} \\
\tau & \mapsto & \sigma \\
\end{array}
\]

with \( \#P-LRmax(\tau) = \#\text{cycles}(\sigma) \)
Foata's thesis and disjoint unions of chains

Can we have the best of both worlds, i.e.

\[ \text{Poin}(K_P,t) = \sum_{\sigma \in \text{LinExt}(P)} t^{\# \text{cycles}(\sigma)} \]

for some natural notion of "cycles" when \( \sigma \in \text{LinExt}(P) \)?

Amazingly, the answer is YES when \( P \) is a disjoint union of chains, where Foata's 1965 thesis can be re-interpreted as giving a natural factorization into cycles for elements of \( \text{LinExt}(P) \).
Labeling disjoint unions of chains two ways

\[ P_{(2,3,2,3)} = \begin{array}{cccc}
5 & 10 & 1 & 1 \\
4 & 7 & 9 & 1 \\
1 & 1 & 1 & 1 \\
1 & 3 & 6 & 8
\end{array} \quad \leftrightarrow \quad \begin{array}{cccc}
2 & 1 & 4 & 1 \\
1 & 2 & 3 & 4 \\
1 & 1 & 1 & 1 \\
1 & 2 & 3 & 4
\end{array} \]

STANDARD LABELS

MULTISET LABELS

gives an easy bijection

LinExt(P_{a}) \leftrightarrow \text{permutations of the multiset } a_1 a_2 a_3 a_4 \ldots

38961427105 \leftrightarrow 2443121342
Foata defined intercalation product on multiset permutations in 2-line notation:

\[
\begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \, \times \, \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ 2 & 4 & 3 & 1 & 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 3 & 4 & 4 \\ 2 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix}
\]

And then he showed they have an essentially unique factorization into (ordinary) cycles, e.g.

\[
\begin{pmatrix} 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 & 4 \\ 2 & 4 & 4 & 3 & 1 & 2 & 1 & 3 & 4 & 2 \end{pmatrix}
\]

\[
= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \, T \, \begin{pmatrix} 1 & 1 & 2 & 2 & 3 & 4 & 4 \\ 2 & 4 & 3 & 1 & 1 & 4 & 2 \end{pmatrix}
\]

\[
= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 2 & 3 \end{pmatrix} \, T \left( \begin{pmatrix} 4 \\ 4 \end{pmatrix} \, T \left( \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \, T \left( \begin{pmatrix} 1 & 2 & 4 \\ 4 & 1 & 2 \end{pmatrix} \right) \right) \right)
\]

\[\text{allowed to swap these two, because they commute}\]
THEOREM (Dorpalen-Barry - Kim - R. 2019)

When $P_a$ is a disjoint union of chains of sizes $a = (a_1, a_2, \ldots, a_n)$, then

$$\text{Poin}(P_a, t) = \sum_{\sigma \in \text{LinExt}(P_a)} t^{\#_{\text{Pfaff\text{-cycles}}}^{\text{Foata\text{-cycles}}} \sigma}$$

where $\#_{\text{Pfaff}\text{-cycles}}^{\text{Foata\text{-cycles}}} \sigma$ means the number of cycles in Foata's unique decomposition for the permutation of the multiset $1^{a_1} 2^{a_2} 3^{a_3} \ldots$ corresponding to $\sigma$. 
**Example** \( a = (2,2) \)

\[
\begin{array}{ccc}
\sigma & \text{Permuation of } 1^2 2^2 & \# \text{Foot cycles}(\sigma) \\
1234 & (1 1 2 2) = (1) \tau(1) \tau(2) \tau(2) & 4 \ \{ \ c_0 = 1 \} \\
1324 & (1 2 1 2) = (1) \tau(1) \tau(2) \tau(2) & 3 \ \{ \ c_1 = 4 \} \\
1342 & (1 2 2 1) = (1) \tau(2) \tau(1) \tau(2) & 3 \\
3124 & (2 1 2 1) = (1 2) \tau(1) \tau(2) & 3 \\
3142 & (2 1 2 1) = (2) \tau(1 2) \tau(1) & 3 \\
3412 & (2 2 1 1) = (1 2) \tau(2 1) & 2 \ \{ \ c_2 = 1 \}
\end{array}
\]
Foata used his theory for generating functions, including a new proof of MacMahon's Master Theorem.

We used it to get this generating function for $\text{Poin}(P_a,t)$'s:

\[ \sum_{a} \text{Poin}(P_a,t) x_1 x_2 \cdots = \frac{1}{1 - \sum_{j \geq 1} e_j(x) (t-1)(x t-1) \cdots ((j-1)t-1)} \]

where $e_j(x) = j^{\text{th}}$ elementary symmetric function

in $x_1, x_2, \ldots$
QUESTION:

Is there a Foata-style factorization theory for $\text{LinExt}(P)$ of all posets $P$, not just disjoint unions of chains?
Thanks for your attention!