Math 4606    Test 3 Solutions    April, 27, 2001.

Solutions by Aniko and Tamas Wiandt

(1) (25 pts.) Let the function \( f \) mapping \( \mathbb{R}^1 \) into \( \mathbb{R}^1 \) be differentiable at the
point \( x \in \mathbb{R}^1 \) and let \( f'(x) \) denote its derivative. Prove that
\[
\lim_{|h| \to 0} \frac{|f(x + h) - f(x) - f'(x)h|}{|h|} = 0.
\]

**Solution.** If \( f \) is differentiable at \( x \) and the derivative is denoted by \( f'(x) \), then
\[
f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.
\]
This implies that
\[
\lim_{h \to 0} \left( \frac{f(x + h) - f(x)}{h} - f'(x) \right) = 0
\]
and then
\[
\lim_{h \to 0} \frac{f(x + h) - f(x) - f'(x)h}{h} = 0
\]
which implies
\[
\lim_{|h| \to 0} \frac{|f(x + h) - f(x) - f'(x)h|}{|h|} = 0
\]
and the proof is complete.

(2) (25 pts.) Let the function \( f \) map \( \mathbb{R}^2 \) into \( \mathbb{R}^1 \). Define that \( f'(x) \) is a
derivative of the function \( f \) at the point \( x \in \mathbb{R}^2 \).

**Solution.** Use the formula from the previous problem: \( x \in \mathbb{R}^2, h \in \mathbb{R}^2, f'(x) \in \mathbb{R}^2 \) and then \( f'(x) \) is the derivative at \( x \) if
\[
\lim_{|h| \to 0} \frac{|f(x + h) - f(x) - f'(x) \cdot h|}{|h|} = 0.
\]
(3) (25 pts.) State and sketch the proof of the Mean Value theorem.

**Solution.** MVT: If \( f \) is continuous on an interval \([a, b]\) and \( f'(x) \) exists for \( a < x < b \), then there exists \( c \in (a, b) \) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a}.
\]

Proof: Consider the function \( g(x) = f(x) + \frac{f(b) - f(a)}{b - a}x \). This is clearly continuous on \([a, b]\), differentiable on \((a, b)\),

\[
g(a) = g(b) = \frac{f(b)a - bf(a)}{a - b}
\]

so by Rolle’s Theorem there exists \( c \in (a, b) \) such that \( g'(c) = 0 \).

\[
g'(c) = f'(c) + \frac{f(b) - f(a)}{a - b}
\]

and the statement follows.

(4) (25 pts.) Suppose the function \( f \) mapping \( \mathbb{R}^2 \) into \( \mathbb{R}^1 \) has continuous partial derivatives and at the point \( x = (x_1, x_2) \in \mathbb{R}^2 \) define

\[
Df(x) = \left( \frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2} \right).
\]

Prove that \( Df(x) \) is a derivative of the function \( f \) at the point \( x \in \mathbb{R}^2 \).

**Solution.** By Problem (2) the definition of the derivative \( f'(x) \) is

\[
\lim_{|h| \to 0} \frac{|f(x + h) - f(x) - f'(x) \cdot h|}{|h|} = 0.
\]

We have to prove that this statement is true for the above defined \( Df(x) \). Let \( h = (h_1, h_2) \), then

\[
f(x + h) - f(x) = f(x_1 + h_1, x_2 + h_2) - f(x_1, x_2) = f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) + f(x_1 + h_1, x_2) - f(x_1, x_2).
\]

By the MVT

\[
f(x_1 + h_1, x_2 + h_2) - f(x_1 + h_1, x_2) = \frac{\partial f(x_1 + h_1, x_2^*)}{\partial x_2} h_2
\]

for some \( x_2^* \in (x_2, x_2 + h_2) \) and

\[
f(x_1 + h_1, x_2) - f(x_1, x_2) = \frac{\partial f(x_1^*, x_2)}{\partial x_1} h_1
\]

for some \( x_1^* \in (x_1, x_1 + h_1) \).
This implies that
\[
f(x + h) - f(x) - Df(x) \cdot h =
\frac{\partial f(x_1 + h_1, x_2^*)}{\partial x_2} h_2 + \frac{\partial f(x_1^*, x_2)}{\partial x_1} h_1 - (\frac{\partial f(x)}{\partial x_1}, \frac{\partial f(x)}{\partial x_2}) \cdot (h_1, h_2) =
\]
\[
= \left( \frac{\partial f(x_1 + h_1, x_2^*)}{\partial x_2} - \frac{\partial f(x_1^*)}{\partial x_2} \right) h_2 + \left( \frac{\partial f(x_1^*, x_2)}{\partial x_1} - \frac{\partial f(x_1^*)}{\partial x_1} \right) h_1.
\]
Now clearly \(0 \leq |h_1|/|h| \leq 1\) and \(0 \leq |h_2|/|h| \leq 1\) and then by the triangle inequality we obtain that
\[
0 \leq \left| \frac{f(x + h) - f(x) - Df(x) \cdot h}{|h|} \right| \leq \left| \frac{\partial f(x_1 + h_1, x_2^*)}{\partial x_2} - \frac{\partial f(x_1^*)}{\partial x_2} \right| + \left| \frac{\partial f(x_1^*, x_2)}{\partial x_1} - \frac{\partial f(x_1^*)}{\partial x_1} \right|.
\]
The partial derivatives are continuous at \(x = (x_1, x_2)\), so by the squeeze theorem we obtain that
\[
\lim_{|h| \to 0} \left| \frac{f(x + h) - f(x) - Df(x) \cdot h}{|h|} \right| = 0
\]
and we established that \(f'(x) = Df(x)\) and the proof is complete.