Math 5525 – Homework I – Solutions to selected problems.

2c. $x' = \cos x$. The phase portrait has equilibria at the points where $\cos x = 0$, that is, at $x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$, In between these points, there are arrows pointing left or right corresponding to regions where $\cos x < 0$ and $\cos x > 0$, respectively. To find the solution with $x(0) = 0$, use the method of separation of variables to get

$$\int_0^t \frac{dt}{\cos(x(t))} = \int_0^t dt = t.$$

Use $x(t)$ as a change of variables on the left to get

$$t = \int_{x(0)}^{x(t)} \frac{dx}{\cos x} = \ln|\sec x(t) + \tan x(t)| - \ln|\sec 0 + \tan 0|.$$

The second term is 0. Taking the exponential of both sides and using some trigonometry, we need to solve

$$e^t = \frac{|1 + \sin x(t)|}{|\cos x(t)|} = \frac{|1+s|}{\sqrt{1-s^2}} \qquad s = \sin x(t).$$

Squaring both sides gives a quadratic equation for s which leads to

$$\sin x(t) = \frac{e^{2t}-1}{e^{2t}+1} = \frac{e^t - e^{-t}}{e^t + e^{-t}} = \tanh(t).$$

Since the phase portrait shows that $-\frac{\pi}{2} < x(t) < \frac{\pi}{2}$ we can take the inverse sine to get

$$x(t) = \sin^{-1}(\tanh(t)).$$

10.a. $x' = x + \cos t$ is a linear equation in one variable so we can solve it using the method of variation of parameters. First solve the homogeneous equation $x' = x$ to get $x = ce^t$. Then try to solve the nonhomogeneous problem using $x(t) = u(t)e^t$. We have

$$x'(t) = u'(t)e^t + u(t)e^t \qquad x(t) + \cos t = u(t)e^t + \cos t.$$

Setting these equal to each other gives $u'(t) = \cos t\, e^{-t}$. Integrating gives

$$u(t) = c + \frac{1}{2}e^{-t}(\sin t - \cos t) \qquad x(t) = ce^t + \frac{1}{2}(\sin t - \cos t).$$

b. We have $x(0) = x(2\pi)$ if and only if $c = 0$ which gives the periodic solution $\frac{1}{2}(\sin t - \cos t)$. Also, if $c \neq 0$ the solution has $x(t) \to \infty$ as $t \to \infty$ which is impossible for a periodic function, no matter what the period.

c. To find the Poincaré map we need to express the constant c in term of the initial value $x(0)$. We find $c = x(0) + \frac{1}{2}$. Then evaluate at $t = 2\pi$ to get the Poincaré map:

$$x(2\pi) = p(x(0)) = (x(0) + \frac{1}{2})e^{2\pi} + \frac{1}{2}(\sin 2\pi - \cos 2\pi) = (x(0) + \frac{1}{2})e^{2\pi} - \frac{1}{2}.$$

To find the fixed points, set $p(x(0)) = x(0)$. The only solution is $x(0) = -\frac{1}{2}$ which agrees with the initial condition for the solution in part b.

16. $x' = f(x,0) = x^2 - 1 - \cos t$. Using Poincaré maps, we can find at least one periodic solution of period 2π in the interval $(-2,0)$ and another in the interval $(0,2)$. First we show that every solution which starts with $x(0) \in [-2,0]$ has $x(2\pi) \in (-2,0)$. There is a continuous Poincaré map $p : [-2,0] \to (-2,0)$ following solutions from time $t = 0$ to $t = -2\pi$. Since p maps $[-2,0]$ into itself, it has at least one fixed point and it must be in $(-2,0)$. We just need to check the slope field on the boundary of the strip $0 \le t \le 2\pi$, $-2 \le x \le 0$. When $x = -2$ we have $f(-2,t) = 3 - \cos t \ge 2 > 0$ so solutions on this wall are moving into the strip as time increases. When $x = 0$ we have $f(0,t) = -1 - \cos t \le 0$ with equality only when $t = \pi$. All solutions on this wall are moving into the strip, except possible at the point $(t,x) = (\pi,0)$. To check this last point we need to look at higher derivatives of the solution with this initial condition to see that the function $x(t)$ decreases. For any solution we have $x'(t) = f(t,x(t)) = x(t)^2 - 1 - \cos t$. Differentiating again twice gives

$$x''(t) = 2x(t)x'(t) + \sin t \qquad x'''(t) = 2x(t)x''(t) + 2x'(t)^2 + \cos t.$$

The relevant solution has $x(\pi) = x'(\pi) = 0$ and substituting these we find $x''(\pi) = 0$ and $x'''(\pi) = \cos \pi = -1$. From this if follows that for times $t \in (\pi, \pi + \delta)$, $\delta > 0$ sufficiently small, we have $x(t) < 0$. In other words, the solution enters the strip, as desired. (The precise argument for why $x(t) < 0$ makes use of the Taylor series:

$$x(t) = x(\pi) + x'(\pi)(t-\pi) + \frac{1}{2}x''(\pi)(t-\pi)^2 + \frac{1}{6}x'''(\pi)(t-\pi)^3 + r(t)$$

where the remainder $r(t)$ is of order $(t-\pi)^4$. In our case we have $x(t) = -\frac{1}{6}(t-\pi)^3 + r(t)$ and this will be negative for $t \in (\pi, \pi+\delta)$, $\delta > 0$ sufficiently small.)

To find the other fixed point we use a backward Poincaré map from $t = 2\pi$ to $t = 0$. In the strip $0 \le t \le 2\pi$, $0 \le x \le 2$ we already know that the slopes $f(0,t) \le 0$ and we have analyzed the point $(t,x) = (\pi,0)$. As time increases, the solutions leave the strip, but as time decreases, they enter it. On the other wall we have $f(2,t) = 3 - \cos t \ge 2$ so again, the solutions are leaving in forward time, but entering in backward time. So the backward Poincaré map will map $[0,2]$ into $(0,2)$ and must therefore have a fixed point in $(0,2)$.